Web-based Supplementary materials for: "Semiparametric Analysis of Linear Transformation Models with Covariate Measurement Errors"

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This supplementary material contains the proof of the main theorem along with the necessary lemmas, two tables from the simulation study with sample size $n = 200$ and the code for computation.

W-A1 List of Regularity Conditions

(C1) The kernel function $K(\cdot)$ is symmetric, has compact support and is Lipschitz continuous on its support. It satisfies

$$\int K(u)du = 1, \ \int uK(u)du = 0, \ 0 \neq \int u^2 K(u)du < \infty.$$

(C2) The probability density function of $f_U(u)$ is bounded away from zero and infinity and has continuous second derivative.

(C3) The monotone transformation function $H(t)$ is defferentiable.

(C4) The hazard function of the error process $\lambda(\bullet)$ is differentiable.

(C5) The bandwidth h satisfies $nh^2 \to \infty$ and $nh^4 \to 0$ when $n \to \infty$.

(C6) The eigenvalues of Σ_* in (S3) and Σ_1 in (S4) are bounded away from zero and infinity.

W-A2 Lemmas and Their Proofs

Proof of Lemma 1.

Proof of Part i): Consider the estimating equation for H with $\boldsymbol{\theta}_0$ and f_U,

$$\frac{1}{n}\sum_{i=1}^n \Big[dN_i(t) - Y_i(t)d\Lambda_T\{t|W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\}\Big]$$
$$= \frac{1}{n}\sum_{i=1}^n \Big[dN_i(t) - Y_i(t)J\{t|W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\}\hat{H}_t(t,\beta,\boldsymbol{\theta}_0, f_U)dt\Big] = 0.$$

Now taking derivative of both sides of the above expression with respect to $\boldsymbol{\beta}$ we obtain

$$\frac{1}{n}\sum_{i=1}^{n} Y_i(t)\Bigg\{ J\{t|W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\}\hat{H}_{\beta t}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)$$
$$+\left[\frac{\partial}{\partial \boldsymbol{\beta}^\dagger} J\{t|W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}^\dagger, \boldsymbol{\theta}_0, f_U\}\right]_{\boldsymbol{\beta}^\dagger=\boldsymbol{\beta}} \hat{H}_t(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)$$
$$+\Bigg(\int \left[\dot{\lambda}\{\boldsymbol{\beta}_1^T\mathbf{Z}_i + \beta_2 x + \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)\} - \lambda^2\{\boldsymbol{\beta}_1^T\mathbf{Z}_i + \beta_2 x + \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)\}\right]$$
$$\times G\{x|t, W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\}dx$$
$$+J^2\{t|W_i, \mathbf{Z}_i, \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\}\Bigg)\hat{H}_\beta(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)\hat{H}_t(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)\Bigg\} = 0. \quad \text{(S1)}$$

Letting $n \to \infty$ and setting $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ we obtain

$$C_D(t)\frac{\partial}{\partial t}\boldsymbol{\gamma}_1(t) + E[Y(t)\frac{\partial}{\partial \boldsymbol{\beta}} J\{t|W, \mathbf{Z}, H_0(t), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}]\dot{H}_0(t) + C_N(t)\boldsymbol{\gamma}_1(t)\dot{H}_0(t) = 0.$$

This is a first order linear differential equation, and can be equivalently written as

$$\frac{d[\lambda^*\{H_0(t)\}\boldsymbol{\gamma}_1(t)]}{dt} = -\frac{\lambda^*\{H_0(t)\}E[Y(t)\partial J\{t|W, \mathbf{Z}, H_0(t), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}/\partial \boldsymbol{\beta}]}{C_D(t)}\dot{H}_0(t).$$

This yields

$$\boldsymbol{\gamma}_1(t) = -\frac{1}{\lambda^*\{H_0(t)\}}\int_0^t \frac{\lambda^*\{H_0(s)\}}{C_D(s)} E[Y(s)\frac{\partial}{\partial \beta_0} J\{s|W, \mathbf{Z}, H_0(s), \beta_0, \boldsymbol{\theta}_0, f_U\}]dH_0(s),$$

where

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\beta}_0} J\{s|W, \mathbf{Z}, H_0(s), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \\
=\; & \int [\dot{\lambda}\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}x + H_0(s)\} - \lambda^2\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}x + H_0(s)\}]\begin{pmatrix}\mathbf{Z}\\ x\end{pmatrix} G(x|s, W, \mathbf{Z}, H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)dx \\
& + J\{s|W, \mathbf{Z}, H_0(s), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}\int \lambda\{\boldsymbol{\beta}_{10}^T Z + \beta_{20}x + H_0(s)\}\begin{pmatrix}\mathbf{Z}\\ x\end{pmatrix} G(x|s, W, \mathbf{Z}, H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)dx \\
=\; & \int \Big[\dot{\lambda}\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}^T x + H_0(s)\} - \lambda^2\{\boldsymbol{\beta}_{10}^T Z + \beta_{20}x + H_0(s)\} + J\{s|W, \mathbf{Z}, H_0(s), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \\
& \times \lambda\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}^T x + H_0(s)\}\Big]\begin{pmatrix}\mathbf{Z}\\ x\end{pmatrix} G(x|s, W, \mathbf{Z}, H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)dx.
\end{aligned}$$

Hence we obtain the desired result.

Proof of part ii): From equation (S1) we can write

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\partial \hat{H}(t,\boldsymbol{\beta},\boldsymbol{\theta}_0, f_U)}{\partial \boldsymbol{\beta}\partial t}\,|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0} = \boldsymbol{\gamma}_2(t) + o_p(1) \\
=\; & -\frac{E[Y(t)\partial J\{t|W_i, \mathbf{Z}_i, H_0(t), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}/\partial \boldsymbol{\beta}_0] + C_N(t)\boldsymbol{\gamma}_1(t)}{C_D(t)}\dot{H}_0(t) + o_p(1)
\end{aligned}$$

where the expression for $\boldsymbol{\gamma}_1(t)$ is given in Equation (5) of our article. □

Lemma 2. *Let $y_i = Y_i(t)$. Assume $nh^4 \to 0$ when $n \to \infty$. For any function $\mathbf{a}(x, y, w, \mathbf{z}, t)$, we have the expansion*

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\int \mathbf{a}(x, y_i, w_i, \mathbf{z}_i, t)\{\widehat{f}_U(w_i - x) - f_U(w_i - x)\}dx \\ = \; & n^{-1/2}\sum_{i=1}^{n} E\left[\mathbf{a}\{W - v_i, Y(t), W, \mathbf{Z}, t\} - \int \mathbf{a}\{x, Y(t), W, \mathbf{Z}, t\} f_U(W - x)dx\right] + o_p(1). \end{aligned}$$

Proof: Use O_i to denote the ith random observation and o_i its realization. Write $\mathbf{f}(O_i, O_j) = \int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)[K\left\{h^{-1}(v_j - W_i + x)\right\}/h - f_U(W_i - x)]dx$ and $\mathbf{g}(O_i, O_j) = \{\mathbf{f}(O_i, O_j) + \mathbf{f}(O_j, O_i)\}/2$, then $\mathbf{g}$ is a symmetric kernel. Plugging in the form of $\widehat{f}_U(\cdot)$, we have

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)\{\widehat{f}_U(W_i - x) - f_U(W_i - x)\}dx \\ = \; & n^{-3/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)\{K\left(\frac{v_j - W_i + x}{h}\right)/h - f_U(W_i - x)\}dx \\ = \; & n^{-3/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{f}(O_i, O_j) \\ = \; & n^{-3/2}\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{f}(O_i, O_j) + \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{f}(O_j, O_i)\}/2 \\ = \; & n^{-3/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{g}(O_i, O_j) \\ = \; & 2n^{-1/2}\sum_{i=1}^{n} E\{\mathbf{g}(O_i, O_j) \mid O_i\} + \left[n^{-3/2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{g}(O_i, O_j) - 2n^{-1/2}\sum_{i=1}^{n} E\{\mathbf{g}(O_i, O_j) \mid O_i\}\right] \\ = \; & n^{-1/2}\sum_{i=1}^{n} E\{\mathbf{f}(O_i, O_j) \mid O_i\} + n^{-1/2}\sum_{i=1}^{n} E\{\mathbf{f}(O_i, O_j) \mid O_j\} - \sqrt{n}E\{\mathbf{f}(O_i, O_j)\} + o_p(1), \end{aligned}$$

where in the last equality, we used the U-statistic property. We now calculate each term. When $i \neq j$,

$$\begin{aligned} & E\{\mathbf{f}(O_i, O_j) \mid O_i\} \\ = \; & \int\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)\{K\left(\frac{v_j - W_i + x}{h}\right)/h - f_U(W_i - x)\}dx f_U(v_j)dv_j \\ = \; & \int\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)h^{-1}K\left(\frac{v_j - W_i + x}{h}\right) f_U(v_j)dv_j dx - \int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)f_U(W_i - x)dx \\ = \; & \int\int \mathbf{a}(W_i - v_j + hs, y_i, W_i, \mathbf{Z}_i, t)K(s)f_U(v_j)dv_j ds - \int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)f_U(W_i - x)dx \\ = \; & \int \mathbf{a}(W_i - v_j, y_i, W_i, \mathbf{Z}_i, t)f_U(v_j)dv_j - \int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t)f_U(W_i - x)dx + O(h^2) = O(h^2). \end{aligned}$$

Thus, as long as $nh^4 \to 0$, the first term is

$$n^{-1/2}\sum_{i=1}^{n} E\{\mathbf{f}(O_i, O_j) \mid O_i\} = O(n^{1/2}h^2 + n^{-1/2}) = o_p(1).$$

Similarly, when $i \neq j$,

$$\begin{aligned}
& E\{\mathbf{f}(O_i, O_j) \mid O_j\} \\
= & \int\int \mathbf{a}(x, y, W, \mathbf{Z}, t)\{K\left(\frac{v_j - W + x}{h}\right)/h - f_U(W-x)\}dx f_{W,\mathbf{Z},Y}(W, z, y)dWd\mathbf{Z}dy \\
= & \int \mathbf{a}(W - v_j + hs, y, W, \mathbf{Z}, t)K(s)f_{W,\mathbf{Z},Y}(W, z, y)dsdWd\mathbf{Z}dy \\
& - \int \mathbf{a}(x, y, W, \mathbf{Z}, t)f_U(W-x)f_{W,\mathbf{Z},Y}(W, z, y)dxdWd\mathbf{Z}dy \\
= & \int \mathbf{a}(W - v_j, y, W, \mathbf{Z}, t)f_{W,\mathbf{Z},Y}(W, z, y)dWd\mathbf{Z}dy \\
& - \int \mathbf{a}(x, y, W, \mathbf{Z}, t)f_U(W-x)f_{W,\mathbf{Z},Y}(W, z, y)dxdWd\mathbf{Z}dy + O(h^2) \\
= & E\left[\mathbf{a}\{W - v_j, Y(t), W, \mathbf{Z}, t\} - \int \mathbf{a}\{x, Y(t), W, \mathbf{Z}, t\}f_U(W-x)dx\right] + O(h^2).
\end{aligned}$$

Thus, the second term is

$$\begin{aligned}
& n^{-1/2}\sum_{j=1}^{n} E\{\mathbf{f}(O_i, O_j) \mid O_j\} \\
= & n^{-1/2}\sum_{i=1}^{n} E\left[\mathbf{a}\{W - v_i, Y(t), W, \mathbf{Z}, t\} - \int \mathbf{a}\{x, Y(t), W, \mathbf{Z}, t\}f_U(W-x)dx\right] + o_p(1).
\end{aligned}$$

Finally, the third term is obviously of order $o_p(1)$. Combining the three terms, we obtain the desired results. □

<u>Lemma</u> 3. *Assume $\widehat{\boldsymbol\theta}$ is estimated through maximizing (2). Throughout the text, let* $f'_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta) = \partial f_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta)/\partial\boldsymbol\theta$,

$$\begin{aligned}
\mathbf{S}_{X,\mathbf{Z},\boldsymbol\theta}(x, \mathbf{Z}, \boldsymbol\theta) &= \frac{f'_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta)}{f_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta)} \\
\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W, \mathbf{Z}, \boldsymbol\theta) &= \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta)f_U(W-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta)f_U(W-x)dx} = E\{\mathbf{S}_{X,\mathbf{Z},\boldsymbol\theta}(x, \mathbf{Z}, \boldsymbol\theta) \mid W, \mathbf{Z}\} \\
\mathbf{A}_{W,\mathbf{Z}} &= E\left[\frac{\partial}{\partial\boldsymbol\theta^{\mathrm{T}}}\left\{\frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta_0)f_U(W-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z})f_U(W-x)dx}\right\}\right] = E\left\{\frac{\partial\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W,\mathbf{Z},\boldsymbol\theta_0)}{\partial\boldsymbol\theta^{\mathrm{T}}}\right\}.
\end{aligned}$$

In addition, throughout the text, we omit the parameter when that parameter assumes the true value whenever the meaning is clear. If $nh^8 \to 0$ and $nh^2 \to \infty$, then

$$\sqrt{n}(\widehat{\boldsymbol\theta} - \boldsymbol\theta_0) = -\frac{\mathbf{A}_{W,\mathbf{Z}}^{-1}}{\sqrt{n}}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W_i, \mathbf{Z}_i, \boldsymbol\theta_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W, \mathbf{Z}, \boldsymbol\theta_0)f_{X,\mathbf{Z}}(W - v_i, \mathbf{Z})dWd\mathbf{Z}\right\} + o_p(1).$$

Proof: The usual Taylor expansion yields

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \widehat{\boldsymbol{\theta}}) \widehat{f}_U(W_i - x) dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \widehat{\boldsymbol{\theta}}) \widehat{f}_U(W_i - x) dx} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \boldsymbol{\theta}_0) \widehat{f}_U(W_i - x) dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \boldsymbol{\theta}_0) \widehat{f}_U(W_i - x) dx} + \{\mathbf{A}_{W,\mathbf{Z}} + o_p(1)\} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0).
\end{aligned}
$$

We have

$$
\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \boldsymbol{\theta}_0) \widehat{f}_U(W_i - x) dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i; \boldsymbol{\theta}_0) \widehat{f}_U(W_i - x) dx} \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i - x) dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i - x) dx} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i - x) dx} \\
& - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i - x) dx \int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i - x) dx\}^2} \\
& + o_p(1) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i|\mathbf{Z}_i)} \\
& - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) \frac{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} + o_p(1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 = & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} \\
& - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) \frac{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} + \mathbf{A}_{W,\mathbf{Z}} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + o_p(1).
\end{aligned}
$$

This yields

$$
\begin{aligned}
\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = & -\mathbf{A}_{W,\mathbf{Z}}^{-1} n^{-1/2} \sum_{i=1}^{n} \Bigg[\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) + \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} \\
& - \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) \frac{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i) \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} \Bigg] + o_p(1).
\end{aligned}
$$

Using Lemma 2, we have

$$
\begin{aligned}
& n^{-1/2} \sum_{i=1}^{n} \int \frac{f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} \{\widehat{f}_U(W_i - x) - f_U(W_i - x)\} dx \\
= & n^{-1/2} \sum_{i=1}^{n} \left\{ \int \frac{f'_{X|\mathbf{Z}}(W - v_i|\mathbf{Z}, \boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} f_{W,\mathbf{Z}}(W, z) dW d\mathbf{Z} - \int \frac{f'_{X|\mathbf{Z}}(x|\mathbf{Z}, \boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} f_U(W - x) f_{W,\mathbf{Z}}(W, z) dx dW d\mathbf{Z} \right\} + o_p(1),
\end{aligned}
$$

and

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\int \frac{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)}\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx \\
= \; & n^{-1/2}\sum_{i=1}^{n}\Bigg\{\int \frac{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) f_{X|\mathbf{Z}}(W-v_i|\mathbf{Z},\boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} f_{W,\mathbf{Z}}(W,z)dWd\mathbf{Z} \\
& -\int \frac{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) f_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} f_U(W-x) f_{W,\mathbf{Z}}(W,\mathbf{Z})dxdWd\mathbf{Z}\Bigg\} + o_p(1).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \sqrt{n}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0) \\
= \; & -\frac{\mathbf{A}_{W,\mathbf{Z}}^{-1}}{\sqrt{n}}\sum_{i=1}^{n}\Bigg[\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) + \frac{\int f'_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol{\theta}_0)\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)} \\
& -\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)\frac{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol{\theta}_0)\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx}{f_{W|\mathbf{Z}}(W_i \mid \mathbf{Z}_i)}\Bigg] + o_p(1) \\
= \; & -\frac{\mathbf{A}_{W,\mathbf{Z}}^{-1}}{\sqrt{n}}\sum_{i=1}^{n}\Bigg\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) + \int \frac{f'_{X|\mathbf{Z}}(W-v_i|\mathbf{Z},\boldsymbol{\theta}_0)-\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X|\mathbf{Z}}(W-v_i|\mathbf{Z})}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} \\
& \times f_{W,\mathbf{Z}}(W,z)dWd\mathbf{Z} - \int \frac{f'_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol{\theta}_0)-\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X|\mathbf{Z}}(x|\mathbf{Z})}{f_{W|\mathbf{Z}}(W \mid \mathbf{Z})} f_U(W-x) f_{W,\mathbf{Z}}(W,\mathbf{Z}) \\
& dxdWd\mathbf{Z}\} + o_p(1) \\
= \; & -\frac{\mathbf{A}_{W,\mathbf{Z}}^{-1}}{\sqrt{n}}\sum_{i=1}^{n}[\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) \\
& + \int \{\mathbf{S}_{X,\mathbf{Z},\boldsymbol{\theta}}(W-v_i,\mathbf{Z},\boldsymbol{\theta}_0) - \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)\} f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z} \\
& -E\{\mathbf{S}_{X,\mathbf{Z},\boldsymbol{\theta}}(x,\mathbf{Z},\boldsymbol{\theta}_0)\} + E\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)\}] + o_p(1) \\
= \; & -\frac{\mathbf{A}_{W,\mathbf{Z}}^{-1}}{\sqrt{n}}\sum_{i=1}^{n}\Bigg\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\Bigg\} + o_p(1).
\end{aligned}
$$

This is the desired results. □

Lemma 4. *Let $y_i = Y_i(t)$. For any function $\mathbf{a}(x,y,W,\mathbf{Z},t)$,*

$$
\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)\left\{f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U) - f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)\right\}dx \\
= \; & E\left(E[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\} \mid W,\mathbf{Z}]\mathbf{S}^{\mathrm{T}}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) - \mathbf{a}\{X,Y(t),W,\mathbf{Z},t\}\mathbf{S}^{\mathrm{T}}_{X,\mathbf{Z},\boldsymbol{\theta}}(X,\mathbf{Z},\boldsymbol{\theta}_0)\right)\mathbf{A}_{W,\mathbf{Z}}^{-1} \\
& \times\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}
\end{aligned}
$$

$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int (E[\mathbf{a}\{W-v_i,Y(t),W,\mathbf{Z},t\}\mid v_i,W,\mathbf{Z}]-E[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\}\mid W,\mathbf{Z}])$$
$$f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1).$$

Proof:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)\left\{f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\widehat{\boldsymbol\theta},\widehat{f}_U)-f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)\right\}dx$$
$$=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}$$
$$-\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)f_U(W_i-x)dx}\}=C_1+C_2+C_3+C_4.$$

Here

$$\begin{aligned}
C_1 &= n^{-1/2}\sum_{i=1}^{n}\{\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}\\
&\quad -\left.\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)\widehat{f}_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}\right\}\\
&= n^{-1/2}\sum_{i=1}^{n}\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t){f'}^{\mathrm{T}}_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta^*)\widehat{f}_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}(\widehat{\boldsymbol\theta}-\boldsymbol\theta_0)\\
&= E\left\{\frac{\int \mathbf{a}(x,Y,W,\mathbf{Z},t){f'}^{\mathrm{T}}_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta_0)f_U(W-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z},\boldsymbol\theta_0)f_U(W-x)dx}\right\}\sqrt{n}(\widehat{\boldsymbol\theta}-\boldsymbol\theta_0)+o_p(1)\\
&= -E\left\{\mathbf{a}(X,Y,W,\mathbf{Z},t)\mathbf{S}^{\mathrm{T}}_{X,\mathbf{Z},\boldsymbol\theta}(X,\mathbf{Z},\boldsymbol\theta_0)\right\}\mathbf{A}^{-1}_{W,\mathbf{Z}}\\
&\quad \times n^{-1/2}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W_i,\mathbf{Z}_i,\boldsymbol\theta_0)-\int \mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W,\mathbf{Z},\boldsymbol\theta_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}+o_p(1).
\end{aligned}$$

$$\begin{aligned}
C_2 &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)\widehat{f}_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}-\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}\right\}\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\widehat{\boldsymbol\theta})\widehat{f}_U(W_i-x)dx}\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)f_U(W_i-x)dx}\{1+o_p(1)\}\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int\frac{\mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)f_{X|\mathbf{Z}}(x|\mathbf{Z}_i,\boldsymbol\theta_0)}{f_{W|\mathbf{Z}}(W_i|\mathbf{Z}_i)}\{\widehat{f}_U(W_i-x)-f_U(W_i-x)\}dx\{1+o_p(1)\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E\left[\frac{\mathbf{a}\{W-v_i, Y(t), W, \mathbf{Z}, t\} f_{X|\mathbf{Z}}(W-v_i|\mathbf{Z})}{f_{W|\mathbf{Z}}(W|\mathbf{Z})} - \int \frac{\mathbf{a}\{x, Y(t), W, \mathbf{Z}, t\} f_{X|\mathbf{Z}}(x|\mathbf{Z})}{f_{W|\mathbf{Z}}(W|\mathbf{Z})} f_U(W-x)dx\right] + o_p(1) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{\int \frac{E\{\mathbf{a}(W-v_i, Y, W, \mathbf{Z}, t) \mid v_i, W, \mathbf{Z}\} f_{X|\mathbf{Z}}(W-v_i|\mathbf{Z}, \boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W|\mathbf{Z})} f_{W,\mathbf{Z}}(W, z) dW d\mathbf{Z}\right. \\
&\left. - \int \frac{E\{\mathbf{a}(x, Y, W, \mathbf{Z}, t) \mid x, W, \mathbf{Z}\} f_{X|\mathbf{Z}}(x|\mathbf{Z}, \boldsymbol{\theta}_0)}{f_{W|\mathbf{Z}}(W|\mathbf{Z})} f_U(W-x) f_{W,\mathbf{Z}}(W, \mathbf{Z}) dx dW d\mathbf{Z}\right\} + o_p(1) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\int E\{\mathbf{a}(W-v_i, Y, W, \mathbf{Z}, t) \mid v_i, W, \mathbf{Z}\} f_{X,\mathbf{Z}}(W-v_i, \mathbf{Z}) dW d\mathbf{Z} - E\{\mathbf{a}(X, Y, W, \mathbf{Z}, t)\}\right] + o_p(1).
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{\frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \widehat{\boldsymbol{\theta}}) \widehat{f}_U(W_i-x)dx}\right. \\
&\quad \left. - \frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \widehat{f}_U(W_i-x)dx}\right\} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx \int {f'}^{\mathrm{T}}_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}^*) \widehat{f}_U(W_i-x)dx}{\left\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \widehat{\boldsymbol{\theta}}) \widehat{f}_U(W_i-x)dx\right\}\left\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \widehat{f}_U(W_i-x)dx\right\}} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \\
&= -E\left[\frac{\int \mathbf{a}\{x, Y(t), W, \mathbf{Z}, t\} f_{X|\mathbf{Z}}(x|\mathbf{Z}) f_U(W-x)dx \int {f'}^{\mathrm{T}}_{X|\mathbf{Z}}(x|\mathbf{Z}, \boldsymbol{\theta}_0) f_U(W-x)dx}{\left\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}) f_U(W-x)dx\right\}^2}\right] \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + o_p(1) \\
&= -E\left(E[\mathbf{a}\{X, Y(t), W, \mathbf{Z}, t\} \mid W, \mathbf{Z}] \mathbf{S}^{\mathrm{T}}_{W,\mathbf{Z},\boldsymbol{\theta}}(W, \mathbf{Z}, \boldsymbol{\theta}_0)\right) \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + o_p(1) \\
&= E\left(E[\mathbf{a}\{X, Y(t), W, \mathbf{Z}, t\} \mid W, \mathbf{Z}] \mathbf{S}^{\mathrm{T}}_{W,\mathbf{Z},\boldsymbol{\theta}}(W, \mathbf{Z}, \boldsymbol{\theta}_0)\right) \mathbf{A}^{-1}_{W,\mathbf{Z}} \\
&\quad \times \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i, \mathbf{Z}_i, \boldsymbol{\theta}_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W, \mathbf{Z}, \boldsymbol{\theta}_0) f_{X,\mathbf{Z}}(W-v_i, \mathbf{Z}) dW d\mathbf{Z}\right] + o_p(1).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
C_4 &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{\frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \widehat{f}_U(W_i-x)dx} - \frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx}{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx}\right\} \\
&= \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\int \mathbf{a}(x, y_i, W_i, \mathbf{Z}_i, t) f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx \int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0)\{\widehat{f}_U(W_i-x) - f_U(W_i-x)\}dx}{\left\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) \widehat{f}_U(W_i-x)dx\right\}\left\{\int f_{X|\mathbf{Z}}(x|\mathbf{Z}_i, \boldsymbol{\theta}_0) f_U(W_i-x)dx\right\}} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int \frac{\int \mathbf{a}(x^*, y_i, W_i, \mathbf{Z}_i, t) f_{X,W|\mathbf{Z}}(x^*, W_i|\mathbf{Z}_i) dx^* f_{X|\mathbf{Z}}(x|\mathbf{Z}_i)}{f^2_{W|\mathbf{Z}}(W_i|\mathbf{Z}_i)} \{\widehat{f}_U(W_i-x) - f_U(W_i-x)\}dx\{1 + o_p(1)\} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} E\left[\frac{\int \mathbf{a}\{x^*, Y(t), W, \mathbf{Z}, t\} f_{X,W|\mathbf{Z}}(x^*, W|\mathbf{Z}) dx^* f_{X|\mathbf{Z}}(W-v_i|\mathbf{Z})}{f^2_{W|\mathbf{Z}}(W|\mathbf{Z})}\right. \\
&\quad \left. - \int \frac{\int \mathbf{a}\{x^*, Y(t), W, \mathbf{Z}, t\} f_{X,W|\mathbf{Z}}(x^*, W|\mathbf{Z}) dx^* f_{X|\mathbf{Z}}(x|\mathbf{Z})}{f^2_{W|\mathbf{Z}}(W|\mathbf{Z})} f_U(W-x)dx\right] + o_p(1)
\end{aligned}
$$

$$= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\int E[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\} \mid W,\mathbf{Z}]f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z} - E\left[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\}\right]\right) + o_p(1).$$

Combining the above results, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int \mathbf{a}(x,y_i,W_i,\mathbf{Z}_i,t)\left\{f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U) - f_{X|W,\mathbf{Z}}(x,W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)\right\}dx \\
= \; & E\left(E[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\} \mid W,\mathbf{Z}]\mathbf{S}^{\mathrm{T}}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) - \mathbf{a}\{X,Y(t),W,\mathbf{Z},t\}\mathbf{S}^{\mathrm{T}}_{X,\mathbf{Z},\boldsymbol{\theta}}(X,\mathbf{Z},\boldsymbol{\theta}_0)\right) \\
&\times \mathbf{A}^{-1}_{W,\mathbf{Z}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\} \\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int\left(E[\mathbf{a}\{W-v_i,Y(t),W,\mathbf{Z},t\} \mid v_i,W,\mathbf{Z}] - E[\mathbf{a}\{X,Y(t),W,\mathbf{Z},t\} \mid W,\mathbf{Z}]\right)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z} \\
&+o_p(1).
\end{aligned}$$

Hence the result is proved. □

Lemma 5. *Let $\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)$ solve $U_H(\beta_0,H,\boldsymbol{\theta}_0,f_U)=0$. Define $\mathbf{a}_i=\mathbf{a}(W_i,\mathbf{Z}_i)$,*

$$\begin{aligned}
&\mathbf{D}_1(\mathbf{a},t) = E\Bigg(Y(t)\mathbf{a}\int[\dot{\lambda}\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\} - \lambda^2\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}] \\
&\quad\times G(x \mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx + Y(t)\mathbf{a}J^2(t \mid W,\mathbf{Z},H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Bigg), \\
&\mathbf{D}_2(\mathbf{a},t) = E\Bigg\{\int E\{Y(t) \mid x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\} - J(t \mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)] \\
&\quad\times G(x \mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\mathbf{S}^{\mathrm{T}}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0) \\
&\quad-\int E\{Y(t) \mid x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\} - J(t \mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)] \\
&\quad\mathbf{S}^{\mathrm{T}}_{X,\mathbf{Z},\boldsymbol{\theta}}(x,\mathbf{Z},\boldsymbol{\theta}_0)G(x \mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\Bigg\}\mathbf{A}^{-1}_{W,\mathbf{Z}}, \\
&\mathbf{Q}_i(\mathbf{a},t) = \mathbf{D}_2(\mathbf{a},t)\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0) - \int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\} \\
&\quad+\int\Bigg(E\{Y(t) \mid W,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\} - J(t \mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)] \\
&\quad\times\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]/\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x \mid W,\mathbf{Z})dx \\
&\quad-\int E\{Y(t) \mid x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\} - J(t \mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)] \\
&\quad G(x \mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\Bigg)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}.
\end{aligned}$$

Then $C_N(t) = \mathbf{D}_1(1,t)$ *and*

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i\left[J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\} - J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right] \\
= \;& \mathbf{D}_1(\mathbf{a},t)\sqrt{n}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U) - \widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\{1+o_p(1)\} + n^{-1/2}\sum_{i=1}^{n}\mathbf{Q}_i(\mathbf{a},t) + o_p(1).
\end{aligned}
$$

Proof: The first part of the lemma is obvious after replacing $\mathbf{a}$ by 1 in $\mathbf{D}_1(a,t)$. From

$$
\begin{aligned}
& J\{t|W,\mathbf{Z},\hat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\} \\
=& \frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z},\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z},\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx},
\end{aligned}
$$

we have

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i\left[J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\} - J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right] \\
= \;& B_1+B_2+B_3+B_4+B_5.
\end{aligned}
$$

Here

$$
\begin{aligned}
& B_1 = n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i \\
& \left(\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}\right. \\
& \left.-\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}\right) \\
& = n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i\int[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\} - \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}] \\
& \quad \times G\{x\mid t,W_i,\mathbf{Z}_i,\widehat{H}(\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}dx \\
& = n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i\int \dot{\lambda}\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}^*,f_U^*)\}G\{x\mid t,W_i,\mathbf{Z}_i,\widehat{H}(\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}dx \\
& \quad \times\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U) - \widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\} \\
& = E\left[Y(t)\mathbf{a}\int \dot{\lambda}\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}G(x\mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\right] \\
& \quad \times\sqrt{n}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U) - \widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\{1+o_p(1)\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
B_2 &= n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i \\
&\Bigg(\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx} \\
&-\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}\Bigg) \\
&= -n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i\Bigg(\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}^*,f_U^*)\} \\
&\qquad \times\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}^*,f_U^*)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx \\
&\qquad /\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx\Bigg) \\
&\qquad \times\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\} \\
&= -E\left[Y(t)\mathbf{a}\int \lambda^2\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}G(x\mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\right] \\
&\qquad \times\sqrt{n}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\{1+o_p(1)\}.
\end{aligned}
$$

Using Lemma 4

$$
\begin{aligned}
B_3 &= n^{-1/2}\sum_{i=1}^{n} Y_i(t)\mathbf{a}_i \\
&\Bigg(\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx} \\
&-\frac{\int \lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)dx}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}\Bigg) \\
&= n^{-1/2}\sum_{i=1}^{n}\int \frac{Y_i(t)\mathbf{a}_i\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx} \\
&\quad \times\Big\{f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)\Big\}dx \\
&= n^{-1/2}\sum_{i=1}^{n}\int \frac{Y_i(t)\mathbf{a}_i\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx} \\
&\quad \times\Big\{f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)\Big\}dx\{1+o_p(1)\} \\
&= E\Bigg(E[\frac{Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mid W,\mathbf{Z}]\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(W,\mathbf{Z},\boldsymbol{\theta}_0)-
\end{aligned}
$$

$$\frac{Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{X,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(X,\mathbf{Z},\boldsymbol{\theta}_0)\Bigg)\mathbf{A}_{W,\mathbf{Z}}^{-1}$$

$$\times n^{-1/2}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)-\int\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}$$

$$+n^{-1/2}\sum_{i=1}^{n}\int\Bigg(E[\frac{Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x|W,\mathbf{Z})dx}|v_i,W,\mathbf{Z}]$$

$$-E[\frac{Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mid W,\mathbf{Z}]\Bigg)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1)$$

$$=E\Bigg(\frac{E[Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]\mid W,\mathbf{Z}]}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(W,\mathbf{Z},\boldsymbol{\theta}_0)-$$

$$\frac{Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{X,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(X,\mathbf{Z},\boldsymbol{\theta}_0)\Bigg)\mathbf{A}_{W,\mathbf{Z}}^{-1}$$

$$\times n^{-1/2}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)-\int\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}$$

$$+n^{-1/2}\sum_{i=1}^{n}\int\Bigg\{\frac{E(Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]|v_i,W,\mathbf{Z})}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x|W,\mathbf{Z})dx}$$

$$-\frac{E(Y(t)\mathbf{a}\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]\mid W,\mathbf{Z})}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\Bigg\}f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1).$$

$$B_4=n^{-1/2}\sum_{i=1}^{n}Y_i(t)\mathbf{a}_i$$

$$\Bigg(\frac{\int\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)dx}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}$$

$$-\frac{\int\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)dx}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx}\Bigg)$$

$$=n^{-1/2}\sum_{i=1}^{n}\int Y_i(t)\mathbf{a}_i\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]$$

$$\times f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0,f_U)dx\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}^*,f_U^*)\}]$$

$$\times\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}^*,f_U^*)\}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx$$

$$\div\Bigg\{\left(\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx\right)$$

$$\times\left(\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}_i+\beta_{20}x+\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}]f_{X|W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol{\theta}},\widehat{f}_U)dx\right)\Bigg\}$$

$$= n^{-1/2}\sum_{i=1}^{n}\frac{Y_i(t)\mathbf{a}_i\left(\int\lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx\right)^2}{\left(\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx\right)^2}$$
$$\{\widehat H(t,\boldsymbol\beta_0,\widehat{\boldsymbol\theta},\widehat f_U)-\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}\{1+o_p(1)\}$$
$$= E\left\{Y(t)\mathbf{a}J^2(t\mid W,\mathbf{Z},H_0,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\right\}\sqrt{n}\{\widehat H(t,\boldsymbol\beta_0,\widehat{\boldsymbol\theta},\widehat f_U)-\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}\{1+o_p(1)\}.$$

Finally,

$$B_5 = n^{-1/2}\sum_{i=1}^{n}Y_i(t)\mathbf{a}_i$$
$$\left(\frac{\int\lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}]f_{X\mid W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)dx}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}]f_{X\mid W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\widehat{\boldsymbol\theta},\widehat f_U)dx}\right.$$
$$\left.-\frac{\int\lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}]f_{X\mid W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)dx}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+\widehat H(t,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\}]f_{X\mid W,\mathbf{Z}}(x|W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)dx}\right)$$
$$= -n^{-1/2}\sum_{i=1}^{n}Y_i(t)\mathbf{a}_i\frac{\int\lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx}{\left(\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx\right)^2}$$
$$\times\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]\{f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol\theta},\widehat f_U)-f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)\}dx\{1+o_p(1)\}$$
$$= -n^{-1/2}\sum_{i=1}^{n}\int\frac{Y_i(t)\mathbf{a}_iJ(t\mid W_i,\mathbf{Z}_i,H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}_i+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i)dx}$$
$$\times\{f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\widehat{\boldsymbol\theta},\widehat f_U)-f_{X\mid W,\mathbf{Z}}(x\mid W_i,\mathbf{Z}_i,\boldsymbol\theta_0,f_U)\}dx\{1+o_p(1)\}$$
$$= -E\left(E[\frac{Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mid W,\mathbf{Z}]\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}^{\mathrm T}(W,\mathbf{Z},\boldsymbol\theta_0)\right.$$
$$\left.-\frac{Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{X,\mathbf{Z},\boldsymbol\theta}^{\mathrm T}(X,\mathbf{Z},\boldsymbol\theta_0)\right)\mathbf{A}_{W,\mathbf{Z}}^{-1}$$
$$\times n^{-1/2}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W_i,\mathbf{Z}_i,\boldsymbol\theta_0)-\int\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}(W,\mathbf{Z},\boldsymbol\theta_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}$$
$$-n^{-1/2}\sum_{i=1}^{n}\int\left(E[\frac{Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mid v_i,W,\mathbf{Z}]\right.$$
$$\left.-E[\frac{Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x|W,\mathbf{Z})dx}|W,\mathbf{Z}]\right)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1)$$
$$= -E\left(\frac{E(Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}X+H_0(t)\}]\mid W,\mathbf{Z})}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{W,\mathbf{Z},\boldsymbol\theta}^{\mathrm T}(W,\mathbf{Z},\boldsymbol\theta_0)\right.$$
$$\left.-\frac{Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol\beta_0,\boldsymbol\theta_0,f_U)\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}X+H_0(t)\}]}{\int\exp[-\Lambda\{\boldsymbol\beta_{10}^{\mathrm T}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X\mid W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\mathbf{S}_{X,\mathbf{Z},\boldsymbol\theta}^{\mathrm T}(X,\mathbf{Z},\boldsymbol\theta_0)\right)\mathbf{A}_{W,\mathbf{Z}}^{-1}$$

$$\times n^{-1/2}\sum_{i=1}^{n}\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)-\int \mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}$$

$$-n^{-1/2}\sum_{i=1}^{n}\int\left(\frac{E(Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]\mid v_i,W,\mathbf{Z})}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx}\right.$$

$$\left.-\frac{E(Y(t)\mathbf{a}J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]|W,\mathbf{Z})}{\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x|W,\mathbf{Z})dx}\right)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1).$$

Combining the above results, we obtain

$$\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}Y_i(t)\mathbf{a}_i\left[J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\}-J\{t|W_i,\mathbf{Z}_i,\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right]\\
= & E\left(Y(t)\mathbf{a}\int[\dot{\lambda}\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}-\lambda^2\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]G(x\mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\right.\\
& \left.+Y(t)\mathbf{a}J^2(t\mid W,\mathbf{Z},H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\right)\sqrt{n}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\{1+o_p(1)\}\\
& +E\left\{\int E\{Y(t)|x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}-J(t|W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]G(x|t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\right.\\
& \times\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(W,\mathbf{Z},\boldsymbol{\theta}_0)-\int E\{Y(t)\mid x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}-J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]\\
& \left.\mathbf{S}_{X,\mathbf{Z},\boldsymbol{\theta}}^{\mathrm{T}}(x,\mathbf{Z},\boldsymbol{\theta}_0)G(x\mid t,W,\mathbf{Z},H_0,\boldsymbol{\theta}_0,f_U)dx\right\}\mathbf{A}_{W,\mathbf{Z}}^{-1}\\
& \times n^{-1/2}\sum_{i=1}^{n}\left[\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)-\int\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right]\\
& +n^{-1/2}\sum_{i=1}^{n}\int\left[\{E(Y(t)\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}-J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]\right.\\
& \times\ \exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]\mid v_i,W,\mathbf{Z})-E(Y(t)\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}\\
& -J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}X+H_0(t)\}]\mid W,\mathbf{Z})\}\\
& \left./\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx\right]f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}+o_p(1)\\
= & \mathbf{D}_1(\mathbf{a},t)\sqrt{n}\{\widehat{H}(t,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)-\widehat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\}\{1+o_p(1)\}\\
& +n^{-1/2}\sum_{i=1}^{n}\mathbf{D}_2(\mathbf{a},t)\left\{\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W_i,\mathbf{Z}_i,\boldsymbol{\theta}_0)-\int\mathbf{S}_{W,\mathbf{Z},\boldsymbol{\theta}}(W,\mathbf{Z},\boldsymbol{\theta}_0)f_{X,\mathbf{Z}}(W-v_i,\mathbf{Z})dWd\mathbf{Z}\right\}\\
& +n^{-1/2}\sum_{i=1}^{n}\int\left(E\{Y(t)\mid W,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}-J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]\right.\\
& \times\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}(W-v_i)+H_0(t)\}]/\int\exp[-\Lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}]f_{X|W,\mathbf{Z}}(x\mid W,\mathbf{Z})dx\\
& -\int E\{Y(t)\mid x,\mathbf{Z}\}\mathbf{a}[\lambda\{\boldsymbol{\beta}_{10}^{\mathrm{T}}\mathbf{Z}+\beta_{20}x+H_0(t)\}-J(t\mid W,\mathbf{Z},H,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)]
\end{aligned}$$

$$
G(x \mid t, W, \mathbf{Z}, H_0, \boldsymbol{\theta}_0, f_U)dx\Bigg) f_{X,\mathbf{Z}}(W - v_i, \mathbf{Z})dWd\mathbf{Z} + o_p(1)
$$

$$
= \mathbf{D}_1(\mathbf{a}, t)\sqrt{n}\{\widehat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \widehat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\}\{1 + o_p(1)\} + n^{-1/2}\sum_{i=1}^{n}\mathbf{Q}_i(\mathbf{a}, t) + o_p(1).
$$

This is the desired results. □

<u>Lemma</u> 6.

$$
\begin{aligned}
&\sqrt{n}\{\hat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\} \\
= \; & -n^{-1/2}\sum_{i=1}^{n}\exp\left\{\int_0^t \frac{-C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2}\right\} \\
&\times\int_0^t \exp\left\{\int_0^s \frac{C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2}\right\}\frac{\mathbf{Q}_i(1, s)E\{dN(s)\}}{(E[Y(s)J\{s|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} + o_p(1) \\
= \; & -n^{-1/2}\sum_{i=1}^{n}\exp\left\{\int_0^t \frac{-C_N(u)dH_0(u)}{C_D(u)}\right\}\int_0^t \exp\left\{\int_0^s \frac{C_N(u)dH_0(u)}{C_D(u)}\right\}\frac{\mathbf{Q}_i(1, s)dH_0(s)}{C_D(s)} + o_p(1).
\end{aligned}
$$

Proof: From the definition of $\hat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U)$ and $\hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)$, we have

$$
\begin{aligned}
&\sqrt{n}\{\hat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\} \\
= \; & \int_0^t \frac{n^{-1/2}\sum_{i=1}^{n} dN_i(u)}{n^{-1}\sum_{i=1}^{n} Y_i(u)J\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U\}} \\
&- \int_0^t \frac{n^{-1/2}\sum_{i=1}^{n} dN_i(u)}{n^{-1}\sum_{i=1}^{n} Y_i(u)J\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}} \\
= \; & -\int_0^t \frac{\{n^{-1}\sum_{i=1}^{n} dN_i(u)\}\{1 + o_p(1)\}}{\left[n^{-1}\sum_{i=1}^{n} Y_i(u)J\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}\right]^2} \\
&\times\left(n^{-1/2}\sum_{i=1}^{n} Y_i(u)\left[J\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U\} - J\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}\right]\right).
\end{aligned}
$$

Using Lemma 5, we have to the first order

$$
\begin{aligned}
&\sqrt{n}\{\hat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\} \\
= \; & \int_0^t \frac{-E\{dN_i(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \\
&\times\left[C_N(u)\sqrt{n}\{\widehat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \widehat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\}\{1 + o_p(1)\} + n^{-1/2}\sum_{i=1}^{n}\mathbf{Q}_i(1, u)\right] + o_p(1).
\end{aligned}
$$

To the leading order, this is an integral equation of the form $y(t) = \int_0^t a(u)y(u)du + b(t)$, which has the solution $y(t) = \exp\{\int_0^t a(u)du\} \int_0^t \exp\{-\int_0^s a(u)du\}b'(s)ds$ when $y(0) = 0$. With

$$a(u) = \frac{-E\{dN_i(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \left[C_N(u)\{1 + o_p(1)\}\right] = \frac{-dH_0(u)}{C_D(u)} \left[C_N(u)\{1 + o_p(1)\}\right]$$

and

$$b(t) = \int_0^t \frac{-E\{dN_i(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \left[n^{-1/2} \sum_{i=1}^n \mathbf{Q}_i(1, u)\right] = \int_0^t \frac{-dH_0(u)}{C_D(u)} \left[n^{-1/2} \sum_{i=1}^n \mathbf{Q}_i(1, u)\right],$$

and inserting y, a and b, we have

$$\begin{aligned}
& \sqrt{n}\{\hat{H}(t, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\} \\
= & -n^{-1/2} \sum_{i=1}^n \exp\left\{\int_0^t \frac{-C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2}\right\} \\
& \times \int_0^t \exp\left\{\int_0^s \frac{C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2}\right\} \frac{\mathbf{Q}_i(1, s)E\{dN(s)\}}{(E[Y(s)J\{s|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} + o_p(1) \\
= & -n^{-1/2} \sum_{i=1}^n \exp\left\{\int_0^t \frac{-C_N(u)dH_0(u)}{C_D(u)}\right\} \int_0^t \exp\left\{\int_0^s \frac{C_N(u)dH_0(u)}{C_D(u)}\right\} \frac{\mathbf{Q}_i(1, s)dH_0(s)}{C_D(s)} + o_p(1).
\end{aligned}$$

□

<u>Lemma</u> 7.

$$\hat{H}(t, \beta_0, \boldsymbol{\theta}_0, f_U) - H_0(t) = \frac{1}{\lambda^*\{H_0(t)\}n} \sum_{i=1}^n \int_0^t \frac{\lambda^*\{H_0(u)\}dM_i(u)}{C_D(u)} + o_p(n^{-1/2}). \tag{S2}$$

Proof: From the definition of $\hat{H}(t, \beta_0, \boldsymbol{\theta}_0, f_U)$ and the Taylor's expansion, we can Write

$$\begin{aligned}
\hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) \;=\; & \int_0^t \frac{\sum_{i=1}^n dN_i(u)}{\sum_{i=1}^n Y_i(u)J\{u|W_i, \mathbf{Z}_i, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}} \\
& - \int_0^t \frac{\{\sum_{i=1}^n dN_i(u)\}[\sum_{i=1}^n Y_i(u)\{\partial J\{u|W_i, \mathbf{Z}_i, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}/\partial H_0\}]}{\{\sum_{i=1}^n Y_i(u)J\{u|W_i, \mathbf{Z}_i, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\}^2} \\
& \times\{\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - H_0(u)\} + o_p\left\{\int_0^t |\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - H_0(u)|du\right\}.
\end{aligned}$$

Replacing $dN_i(u)$ by $Y_i(u)\lambda_T(u|W_i, \mathbf{Z}_i, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)du + dM_i(u)$, and using the strong law of large numbers as $n \to \infty$, we have

$$\begin{aligned}
\hat{H}(t, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) = & \int_0^t dH_0(u) + n^{-1} \sum_{i=1}^n \int_0^t \frac{dM_i(u)}{C_D(u)} - \int_0^t \frac{\dot{H}_0(u)C_N(u)}{C_D(u)}\{\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - H_0(u)\}du \\
& -n^{-1} \sum_{i=1}^n \int_0^t \frac{C_N(u)}{C_D(u)^2}\{\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - H_0(u)\}dM_i(u) + o_p\left\{\int_0^t |\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - H_0(u)|du\right\}.
\end{aligned}$$

The Martingale central limit theorem implies that

$$n^{-1}\sum_{i=1}^{n}\int_0^t \frac{C_N(u)}{C_D(u)^2}\{\hat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-H_0(u)\}dM_i(u)$$

converges to a mean zero random variable with variance of order $n^{-1}\{\int_0^t|\hat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-H_0(u)|^2du\}$, hence is a negligible term. Thus to the leading order, we obtain

$$\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)=\int_0^t dH_0(u)+\frac{1}{n}\int_0^t\frac{\sum_{i=1}^n dM_i(u)}{C_D(u)}-\int_0^t\frac{C_N(u)\dot{H}_0(u)}{C_D(u)}\{\hat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-H_0(u)\}du.$$

Taking derivative and multiple $\lambda^*\{H_0(t)\}$ on both sides, after combining terms, we obtain that to the first order,

$$d[\lambda^*\{H_0(t)\}\{\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-H_0(t)\}]=\lambda^*\{H_0(t)\}\frac{1}{n}\frac{\sum_{i=1}^n dM_i(t)}{C_D(t)}.$$

This gives the result

$$\hat{H}(t,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-H_0(t)=\frac{1}{\lambda^*\{H_0(t)\}n}\sum_{i=1}^n\int_0^t\frac{\lambda^*\{H_0(u)\}dM_i(u)}{C_D(u)}+o_p(n^{-1/2}).$$

□

W-A3 Proof of Theorem 1.

A Taylor expansion of the estimating equation yields

$$E\left[\frac{1}{n}\mathbf{U}_\beta\{\boldsymbol{\beta},\widehat{H}(\cdot,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_u),\boldsymbol{\theta}_0,f_U\}+o_p(1)\right]\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)=-n^{-1/2}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}.$$

Thus, we first consider the asymptotic expansion of $n^{-1/2}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}$.

$$\begin{aligned}0&=n^{-1/2}\mathbf{U}_\beta\{\widehat{\boldsymbol{\beta}},\widehat{H}(\cdot,\widehat{\boldsymbol{\beta}},\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}\\&=E\left[\frac{\partial}{n\partial\boldsymbol{\beta}_0^{\mathrm{T}}}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\theta}_0,f_U\}\right]\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)+\frac{1}{\sqrt{n}}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}+o_p(1)\\&=\Sigma_1\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)+\frac{1}{\sqrt{n}}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}+o_p(1).\end{aligned}$$

Now write

$$\begin{aligned}\frac{1}{\sqrt{n}}\mathbf{U}_\beta\{\boldsymbol{\beta}_0,\widehat{H}(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\}&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\int_0^\tau\begin{pmatrix}\mathbf{Z}_i\\W_i\end{pmatrix}[dN_i(u)-Y_i(u)\lambda_T\{u|W_i,\mathbf{Z}_i;\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\}du]\\&=\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3+\mathbf{A}_4+\mathbf{A}_5+\mathbf{A}_6,\end{aligned}$$

where

$$
\begin{aligned}
\mathbf{A}_1 &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} dM_i(u),\\
\mathbf{A}_2 &= -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)\left[J\{u|W_i,\mathbf{Z}_i;\hat{H}(u,\boldsymbol{\beta}_0),\boldsymbol{\beta}_0\} - J\{u|W_i,\mathbf{Z}_i;H_0(u),\boldsymbol{\beta}_0\}\right] dH_0(u),\\
\mathbf{A}_3 &= -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)J(u|W_i,\mathbf{Z}_i;H_0(u),\boldsymbol{\beta}_0)\{d\hat{H}(u,\boldsymbol{\beta}_0) - dH_0(u)\},\\
\mathbf{A}_4 &= -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)\Big\{J(u|W_i,\mathbf{Z}_i;\hat{H}(u,\boldsymbol{\beta}_0),\boldsymbol{\beta}_0) - J(u|W_i,\mathbf{Z}_i;H_0(u),\boldsymbol{\beta}_0)\Big\}\\
&\quad \times\{d\hat{H}(u,\boldsymbol{\beta}_0) - dH_0(u)\}\\
&= o_p(1)\\
\mathbf{A}_5 &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)J\{u|W_i,\mathbf{Z}_i;\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\\
&\quad d\Big\{\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U) - \widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\Big\}\\
\mathbf{A}_6 &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)\Big[J\{u|W_i,\mathbf{Z}_i;\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\\
&\quad -J\{u|W_i,\mathbf{Z}_i;\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U\}\Big] d\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U).
\end{aligned}
$$

Using the mean-value theorem we can write

$$
\begin{aligned}
&\mathbf{A}_2 = -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix} Y_i(u)\Big[\int \dot{\lambda}\{\boldsymbol{\beta}_{10}^T\mathbf{Z}_i + \beta_{20}x + H_0(u)\}G(x|u,W_i,\mathbf{Z}_i;H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx\\
&-\int \lambda^2\{\boldsymbol{\beta}_{10}^T\mathbf{Z}_i + \beta_{20}x + H_0(u)\}G(x|u,W_i,\mathbf{Z}_i;H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx\\
&+J^2(u|W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Big]\{\hat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U) - H_0(u)\}dH_0(u) + o_p(1).
\end{aligned}
$$

Now replacing $\hat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U) - H_0(u)$ by (S2), and changing the order of the two summations, and applying the strong law of large number we obtain

$$
\begin{aligned}
&\mathbf{A}_2 = -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\frac{\lambda^*\{H_0(s)\}}{C_D(s)}dM_i(s)\\
&\times E\Bigg(\int_s^{\tau} Y(u)\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}\Big[\int \dot{\lambda}\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}x + H_0(u)\}G(x|u,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx\\
&-\int \lambda^2\{\boldsymbol{\beta}_{10}^T\mathbf{Z} + \beta_{20}x + H_0(u)\}G(x|u,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx + J^2(u|W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Big]\frac{dH_0(u)}{\lambda^*\{H_0(u)\}}\Bigg) + o_p(1).
\end{aligned}
$$

Taking derivative of (S2) and using $d\lambda^*\{H_0(u)\} = \lambda^*\{H_0(u)\}C_N(u) \times dH_0(u)/C_D(u)$, We obtain an expression for $d\hat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) - dH_0(u)$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{dM_j(u)}{C_D(u)} - \frac{dH_0(u)}{n\lambda^*\{H_0(u)\}}\frac{C_N(u)}{C_D(u)}\int_0^u\sum_{j=1}^{n}\frac{\lambda^*\{H_0(s)\}dM_j(s)}{C_D(s)}$$

which is then used in $\mathbf{A}_3$, and get

$$\begin{aligned}
\mathbf{A}_3 &= -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix}Y_i(u)J(u|W_i,\mathbf{Z}_i;H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Bigg[\frac{1}{n}\sum_{j=1}^{n}\frac{dM_j(u)}{C_D(u)} \\
&\quad -\frac{dH_0(u)}{n\lambda^*\{H_0(u)\}}\frac{C_N(u)}{C_D(u)}\int_0^u\sum_{j=1}^{n}\frac{\lambda^*\{H_0(s)\}dM_j(s)}{C_D(s)}\Bigg] \\
&= -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\frac{dM_i(u)}{C_D(u)}E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right]+n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\frac{\lambda^*\{H_0(s)\}}{C_D(s)} \\
&\quad \times dM_i(s)\int_s^{\tau}\frac{dH_0(u)C_N(u)}{\lambda^*\{H_0(u)\}C_D(u)}E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right]+o_p(1).
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_5 &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix}Y_i(u)J\{u|W_i,\mathbf{Z}_i;\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}d\left\{\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\right\} \\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix}Y_i(u)J\{u|W_i,\mathbf{Z}_i;H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\} \\
&\quad d\left\{\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\right\}+o_p(1) \\
&= \int_0^{\tau}E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right] \\
&\quad d\sqrt{n}\left\{\widehat{H}(u,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)-\widehat{H}(u,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U)\right\}+o_p(1) \\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right] \\
&\quad d\Bigg[\exp\left\{\int_0^u\frac{-C_N(s)E\{dN(s)\}}{(E[Y(s)J\{s|W,\mathbf{Z},H_0(s),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}])^2}\right\} \\
&\quad \times\int_0^u\exp\left\{\int_0^s\frac{C_N(r)E\{dN(r)\}}{(E[Y(r)J\{r|W,\mathbf{Z},H_0(r),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}])^2}\right\}\frac{\mathbf{Q}_i(1,s)E\{dN_i(s)\}}{(E[Y(s)J\{s|W,\mathbf{Z},H_0(s),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}])^2}\Bigg]+o_p(1) \\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix}Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right] \\
&\quad d\left[\exp\left\{\int_0^u\frac{-C_N(s)dH_0(s)}{C_D(s)}\right\}\int_0^u\exp\left\{\int_0^s\frac{C_N(r)dH_0(r)}{C_D(r)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\right]+o_p(1).
\end{aligned}$$

In $\mathbf{A}_5$, the second to the last equality follows from Lemma 6. Finally, let $\mathbf{Z}_i^* = (\mathbf{Z}_i^{\mathrm{T}}, W_i)^{\mathrm{T}}$.

$$
\begin{aligned}
\mathbf{A}_6 &= n^{-1/2}\sum_{i=1}^n \int_0^\tau \begin{pmatrix} \mathbf{Z}_i \\ W_i \end{pmatrix} Y_i(u) \Big[J\{u|W_i, \mathbf{Z}_i; \widehat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \\
&\quad - J\{u|W_i, \mathbf{Z}_i; \widehat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U\}\Big] d\widehat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) \\
&= -\int_0^\tau \left[\mathbf{D}_1(\mathbf{Z}^*, u)\sqrt{n}\{\widehat{H}(u, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U) - \widehat{H}(u, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)\} + n^{-1/2}\sum_{i=1}^n \mathbf{Q}_i(\mathbf{Z}^*, u) \right] dH_0(u) + o_p(1) \\
&= n^{-1/2}\sum_{i=1}^n \int_0^\tau \Bigg[-\mathbf{Q}_i(\mathbf{Z}^*, t) + \mathbf{D}_1(\mathbf{Z}^*, t)\exp\left\{ \int_0^t \frac{-C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \right\} \\
&\quad \times \int_0^t \exp\left\{ \int_0^s \frac{C_N(u)E\{dN(u)\}}{(E[Y(u)J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \right\} \frac{\mathbf{Q}_i(1, s)E\{dN_i(s)\}}{(E[Y(s)J\{s|W, \mathbf{Z}, H_0(s), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\}])^2} \Bigg] dH_0(t) + o_p(1) \\
&= n^{-1/2}\sum_{i=1}^n \int_0^\tau \Bigg[-\mathbf{Q}_i(\mathbf{Z}^*, t) + \mathbf{D}_1(\mathbf{Z}^*, t)\exp\left\{ \int_0^t \frac{-C_N(u)dH_0(u)}{C_D(u)} \right\} \\
&\quad \times \int_0^t \exp\left\{ \int_0^s \frac{C_N(u)dH_0(u)}{C_D(u)} \right\} \frac{\mathbf{Q}_i(1, s)dH_0(s)}{C_D(s)} \Bigg] dH_0(t) + o_p(1).
\end{aligned}
$$

The second equality in $\mathbf{A}_6$ follows from Lemma 5.

Adding $\mathbf{A}_1, \cdots, \mathbf{A}_6$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{n}} U_\beta\{\boldsymbol{\beta}_0, \hat{H}(\cdot, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \widehat{\boldsymbol{\theta}}, \widehat{f}_U\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^\tau \begin{pmatrix} \mathbf{Z}_i \\ W_i \end{pmatrix} dM_i(u) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^\tau \frac{\lambda^*\{H_0(s)\}}{C_D(s)} dM_i(s) \\
&\quad \times E\Bigg(\int_s^\tau Y(u) \begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} \Bigg[\int \dot{\lambda}\{\boldsymbol{\beta}_{10}^T \mathbf{Z} + \beta_{20}x + H_0(u)\} G(x|u, W, \mathbf{Z}; H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)dx \\
&\quad - \int \lambda^2\{\boldsymbol{\beta}_{10}^T \mathbf{Z} + \beta_{20}x + H_0(u)\} G(x|u, W, \mathbf{Z}; H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U)dx + J^2(u|W, \mathbf{Z}; H_0, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U) \Bigg] \\
&\quad \times \frac{dH_0(u)}{\lambda^*\{H_0(u)\}} \Bigg) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^\tau \frac{dM_i(u)}{C_D(u)} E\left[\begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} Y(u)J\{u|W, \mathbf{Z}; H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \right] \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^\tau \frac{\lambda^*\{H_0(s)\}}{C_D(s)} dM_i(s) \int_s^\tau \frac{dH_0(u)C_N(u)}{\lambda^*\{H_0(u)\}C_D(u)} E\left[\begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} Y(u)J\{u|W, \mathbf{Z}; H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \right] \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^n \int_0^\tau E\left[\begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} Y(u)J\{u|W, \mathbf{Z}; H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \right] d\Bigg[\exp\left\{ \int_0^u \frac{-C_N(s)dH_0(s)}{C_D(s)} \right\} \\
&\quad \times \int_0^u \exp\left\{ \int_0^s \frac{C_N(r)dH_0(r)}{C_D(r)} \right\} \frac{\mathbf{Q}_i(1, s)dH_0(s)}{C_D(s)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^\tau\Bigg[-\mathbf{Q}_i(\mathbf{Z}^*,t)+\mathbf{D}_1(\mathbf{Z}^*,t)\exp\left\{\int_0^t\frac{-C_N(u)dH_0(u)}{C_D(u)}\right\}\\
&\times\int_0^t\exp\left\{\int_0^s\frac{C_N(u)dH_0(u)}{C_D(u)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\Bigg]dH_0(t)\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^\tau\Bigg(\left(\begin{array}{c}\mathbf{Z}_i\\ W_i\end{array}\right)-\frac{\lambda^*\{H_0(u)\}}{C_D(u)}\\
&\times E\Bigg(\int_u^\tau Y(s)\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)\Bigg[\int\dot\lambda\{\boldsymbol{\beta}_{10}^T\mathbf{Z}+\beta_{20}x+H_0(s)\}G(x|s,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx\\
&-\int\lambda^2\{\boldsymbol{\beta}_{10}^T\mathbf{Z}+\beta_{20}x+H_0(s)\}G(x|s,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx+J^2(s|W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Bigg]\\
&\times\frac{dH_0(s)}{\lambda^*\{H_0(s)\}}\Bigg)-\frac{1}{C_D(u)}E\Bigg[\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\Bigg]\\
&+\frac{\lambda^*\{H_0(u)\}}{C_D(u)}\int_u^\tau\frac{dH_0(s)C_N(s)}{\lambda^*\{H_0(s)\}C_D(s)}E\Bigg[\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)Y(s)J\{s|W,\mathbf{Z};H_0(s),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\Bigg]\Bigg)dM_i(u)\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^\tau\Bigg(E\Bigg[\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\Bigg]\exp\left\{\int_0^u\frac{-C_N(s)dH_0(s)}{C_D(s)}\right\}\\
&\times\Bigg[\exp\left\{\int_0^u\frac{C_N(s)dH_0(s)}{C_D(s)}\right\}\frac{\mathbf{Q}_i(1,u)}{C_D(u)}-\frac{C_N(u)}{C_D(u)}\int_0^u\exp\left\{\int_0^s\frac{C_N(l)dH_0(l)}{C_D(l)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\Bigg]\\
&+\Bigg[-\mathbf{Q}_i(\mathbf{Z}^*,u)+\mathbf{D}_1(\mathbf{Z}^*,u)\exp\left\{\int_0^u\frac{-C_N(s)dH_0(s)}{C_D(s)}\right\}\int_0^u\exp\left\{\int_0^s\frac{C_N(l)dH_0(l)}{C_D(l)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\Bigg]\Bigg)dH_0(u)+o_p(1).
\end{aligned}
$$

Therefore, we can write

$$
n^{-1/2}U_\beta\{\boldsymbol{\beta}_0,\hat H(\cdot,\boldsymbol{\beta}_0,\widehat{\boldsymbol{\theta}},\widehat{f}_U),\widehat{\boldsymbol{\theta}},\widehat{f}_U\} \;=\; n^{-1/2}\sum_{i=1}^{n}\int_0^\tau\{\Phi_i(u)dM_i(u)+\Upsilon_i(u)dH_0(u)\}+o_p(1),
$$

where

$$
\begin{aligned}
&\Phi_i(u)=\left(\begin{array}{c}\mathbf{Z}_i\\ W_i\end{array}\right)-\frac{\lambda^*\{H_0(u)\}}{C_D(u)}\\
&\times E\Bigg(\int_u^\tau Y(s)\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)\Bigg[\int\dot\lambda\{\boldsymbol{\beta}_{10}^T\mathbf{Z}+\beta_{20}x+H_0(s)\}G(x|s,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx\\
&-\int\lambda^2\{\boldsymbol{\beta}_{10}^T\mathbf{Z}+\beta_{20}x+H_0(s)\}G(x|s,W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)dx+J^2(s|W,\mathbf{Z};H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\Bigg]\\
&\times\frac{dH_0(s)}{\lambda^*\{H_0(s)\}}\Bigg)-\frac{1}{C_D(u)}E\Bigg[\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\Bigg]\\
&+\frac{\lambda^*\{H_0(u)\}}{C_D(u)}\int_u^\tau\frac{dH_0(s)C_N(s)}{\lambda^*\{H_0(s)\}C_D(s)}E\Bigg[\left(\begin{array}{c}\mathbf{Z}\\ W\end{array}\right)Y(s)J\{s|W,\mathbf{Z};H_0(s),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon_i(u) = \\
&\left[-\mathbf{Q}_i(\mathbf{Z}^*, u)+\mathbf{D}_1(\mathbf{Z}^*, u)\exp\left\{\int_0^u \frac{-C_N(s)dH_0(s)}{C_D(s)}\right\}\int_0^u \exp\left\{\int_0^s \frac{C_N(l)dH_0(l)}{C_D(l)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\right] \\
&+E\left[\begin{pmatrix}\mathbf{Z}\\ W\end{pmatrix} Y(u)J\{u|W,\mathbf{Z};H_0(u),\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U\}\right]\exp\left\{\int_0^u \frac{-C_N(s)dH_0(s)}{C_D(s)}\right\} \\
&\times\left[\exp\left\{\int_0^u \frac{C_N(s)dH_0(s)}{C_D(s)}\right\}\frac{\mathbf{Q}_i(1,u)}{C_D(u)} - \frac{C_N(u)}{C_D(u)}\int_0^u \exp\left\{\int_0^s \frac{C_N(l)dH_0(l)}{C_D(l)}\right\}\frac{\mathbf{Q}_i(1,s)dH_0(s)}{C_D(s)}\right].
\end{aligned}
$$

Observe that $\Phi_i(u)$ is a predictable and bounded process for $u \in (0,\tau]$ with respect to the filtration $\mathcal{F}_{u-} = \sigma\{Y(s), N(s), \mathbf{Z}, W, 0 \le s < u\}$. Due to the martingale property $E\{\int_0^\tau \Phi_i(u)dM_i(u)\} = 0$. On the other hand, $\Upsilon_i(u)$ belongs to a Hilbert space of square integrable random variable with zero mean, i.e., $E\{\Upsilon_i^2(u)\} < \infty$, $E\{\Upsilon_i(u)\} = 0$ for all $u \in (0,\tau]$. Now, using the Martingale central limit theorem we can write $n^{-1/2}U_\beta\{\boldsymbol{\beta}_0, \hat{H}(\cdot, \boldsymbol{\beta}_0, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \widehat{\boldsymbol{\theta}}, \widehat{f}_U\}$ asymptotically follows a normal distribution with mean 0 and variance

$$
\Sigma_* = E\left[\left\{\int_0^\tau \Phi_i(u)\Phi_i^T(u)Y_i(u)\lambda_T(u|W_i,\mathbf{Z}_i,H_0,\boldsymbol{\beta}_0,\boldsymbol{\theta}_0,f_U)\right\}du + \left\{\int_0^\tau \Upsilon_i(u)dH_0(u)\right\}^{\otimes 2}\right]. \tag{S3}
$$

The above equality used the fact that $\mathrm{cov}\{\int_0^\tau \Phi_i(t)dM_i(t), \int_0^\tau \Upsilon_i(t)dH_0(t)\} = 0$. Observe that the randomness of $\Upsilon_i(u)$ comes only from its random covariates $W_i, \mathbf{Z}_i$, and consequently for any $u, u' \in (0,\tau]$, $\mathrm{cov}\{dM_i(u), \Upsilon_i(u')\} = E[E\{dM_i(u)\Upsilon_i(u')|\mathcal{F}_{u-}\}] = E[\Upsilon_i(u')E\{dM_i(u)|\mathcal{F}_{u-}\}] = 0$. Hence,

$$
\begin{aligned}
&\mathrm{cov}\{\int_0^\tau \Phi_i(t)dM_i(t), \int_0^\tau \Upsilon_i(t)dH_0(t)\} = E\left\{\int_0^\tau\int_0^\tau \Phi_i(u)dM_i(u)\Upsilon_i(u')dH_0(u')\right\} \\
&= E\left\{\int_0^\tau\int_0^\tau \Upsilon_i(u')\Phi_i(u)E(dM_i(u)|\mathcal{F}_{u-})dH_0(u')\right\} = 0.
\end{aligned}
$$

We now consider the calculation of Σ_1. Observe that

$$
\begin{aligned}
&\frac{1}{n}\frac{\partial}{\partial\boldsymbol{\beta}^{\mathrm{T}}}\mathbf{U}_\beta\{\boldsymbol{\beta}, \widehat{H}(\cdot,\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U),\boldsymbol{\theta}_0,f_U\} \\
= \; & \frac{1}{n}\sum_{i=1}^n \begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix}\frac{\partial}{\partial\boldsymbol{\beta}^{\mathrm{T}}}\int_0^\tau \left\{dN_i(u) - Y_i(u)J(u|W_i,\mathbf{Z}_i,\widehat{H},\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U)\hat{H}_u(u,\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U)\right\}du \\
= \; & -\frac{1}{n}\sum_{i=1}^n \begin{pmatrix}\mathbf{Z}_i\\ W_i\end{pmatrix}\int_0^\tau Y_i(u)\Big[J_\beta\{u|W_i,\mathbf{Z}_i,\hat{H}(u,\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U),\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U\}\hat{H}_u(u,\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U) \\
& +J(u|W_i,\mathbf{Z}_i,\hat{H},\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U)\hat{H}_{\beta u}(u,\boldsymbol{\beta},\boldsymbol{\theta}_0,f_U)\Big]^{\mathrm{T}} du,
\end{aligned}
$$

where

$$
\begin{aligned}
& J_\beta\{u|W_i, \mathbf{Z}_i, \hat{H}(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\} \\
= & \int \dot{\lambda}\{\boldsymbol{\beta}_1^T \mathbf{Z}_i + \beta_2 x + \hat{H}_0(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)\}[\begin{pmatrix} \mathbf{Z}_i \\ x \end{pmatrix} + \hat{H}_\beta(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)] G\{x|u, W_i, \mathbf{Z}_i, \hat{H}(\cdot, \boldsymbol{\beta}), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\} dx \\
& - \int \lambda^2\{\boldsymbol{\beta}_1^T \mathbf{Z}_i + \beta_2 x + \hat{H}(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)\}[\begin{pmatrix} \mathbf{Z}_i \\ x \end{pmatrix} + \hat{H}_\beta(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)] \\
& \times G\{x|u, W_i, \mathbf{Z}_i, \hat{H}(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\} dx + J\{u|W_i, \mathbf{Z}_i, \hat{H}(\cdot, \boldsymbol{\beta}), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\} \\
& \times \int \lambda\{\boldsymbol{\beta}_1^T \mathbf{Z}_i + \beta_2 x + \hat{H}(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)\}[\begin{pmatrix} \mathbf{Z}_i \\ x \end{pmatrix} + \hat{H}_\beta(u, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U)] \\
& \times G\{x|u, W_i, \mathbf{Z}_i, \hat{H}(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U), \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U\} dx.
\end{aligned}
$$

After setting $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ We obtain

$$
-\frac{1}{n} E \frac{\partial}{\partial \boldsymbol{\beta}^{\mathrm{T}}} U_\beta\{\boldsymbol{\beta}, \hat{H}(\cdot, \boldsymbol{\beta}, \widehat{\boldsymbol{\theta}}, \widehat{f}_U), \widehat{\boldsymbol{\theta}}, \widehat{f}_U\} \mid_{\boldsymbol{\beta}=\boldsymbol{\beta}_0} \stackrel{a.s}{\to} -\frac{1}{n} E \frac{\partial}{\partial \boldsymbol{\beta}^{\mathrm{T}}} U_\beta\{\boldsymbol{\beta}, \hat{H}(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U), \boldsymbol{\theta}_0, f_U\} \mid_{\boldsymbol{\beta}=\boldsymbol{\beta}_0} \stackrel{a.s}{\to} \Sigma_1,
$$

where

$$
\begin{aligned}
\Sigma_1 = \quad & E\Bigg(\int_0^\tau Y(u) \begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} \int \Big[\dot{\lambda}\{\boldsymbol{\beta}_{10}^T Z + \beta_{20} x + H_0(u)\} - \lambda^2\{\boldsymbol{\beta}_{10}^T \mathbf{Z} + \beta_{20} x + H_0(u)\} \\
& + J\{u|W, \mathbf{Z}, H_0(u), \beta_0, \boldsymbol{\theta}_0, f_U\} \lambda\{\boldsymbol{\beta}_{10}^T \mathbf{Z} + \beta_{20} x + H_0(u)\}\Big] \\
& \times \Bigg\{\begin{pmatrix} \mathbf{Z} \\ x \end{pmatrix} + \boldsymbol{\gamma}_1(u)\Bigg\}^T G(x|u, W_i, \mathbf{Z}_i, H_0, \boldsymbol{\beta}, \boldsymbol{\theta}_0, f_U) dx \dot{H}_0(u) du\Bigg) \\
& + E\Big[\int_0^\tau Y(u) \begin{pmatrix} \mathbf{Z} \\ W \end{pmatrix} J\{u|W, \mathbf{Z}, H_0(u), \boldsymbol{\beta}_0, \boldsymbol{\theta}_0, f_U\} \boldsymbol{\gamma}_2^{\mathrm{T}}(u) du\Big].
\end{aligned} \tag{S4}
$$

W-A4 Tables from the simulation study

Table 1: Results of the simulation study where $\log(T) = -Z - X + \epsilon$. The number of replications is 500. NV, CW, and SP stand for the naive, Cheng and Wang's method, and the proposed semiparametric approach. Here SD, MSE, ESE, and CP denote the standard deviation of the estimates, mean squared error, estimated standard error based on the formula, and 95% coverage probability. The sample size was $n = 200$ and $U^* \sim \text{Normal}(0, \sigma_U^2)$ with $\sigma_U^2 = 0.5$.

		10% censoring						50% censoring					
		NV		CW		SP		NV		CW		SP	
r		β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
0	Bias	−0.063	−0.198	0.002	0.026	0.049	0.065	−0.037	−0.146	0.059	0.414	0.043	0.115
	SD	0.276	0.088	0.351	0.145	0.314	0.155	0.367	0.105	0.480	0.281	0.404	0.168
	MSE	0.080	0.047	0.123	0.022	0.101	0.028	0.136	0.032	0.234	0.250	0.165	0.041
	ESE	0.267	0.084			0.320	0.166	0.342	0.096			0.387	0.171
	CP	0.920	0.354			0.972	0.966	0.926	0.63			0.950	0.960
0.5	Bias	−0.019	−0.164	0.021	0.025	0.036	0.061	−0.014	−0.127	0.068	0.291	0.032	0.109
	SD	0.378	0.114	0.388	0.163	0.403	0.171	0.456	0.128	0.524	0.278	0.487	0.186
	MSE	0.143	0.040	0.151	0.027	0.164	0.033	0.208	0.033	0.279	0.162	0.238	0.046
	ESE	0.375	0.115			0.399	0.177	0.439	0.127			0.470	0.183
	CP	0.944	0.690			0.954	0.954	0.938	0.80			0.938	0.948
1	Bias	0.003	−0.152	0.005	0.025	0.038	0.040	0.004	−0.125	0.071	0.227	0.041	0.094
	SD	0.488	0.142	0.459	0.180	0.509	0.196	0.518	0.150	0.607	0.304	0.546	0.208
	MSE	0.238	0.043	0.211	0.033	0.261	0.040	0.268	0.038	0.373	0.144	0.300	0.052
	ESE	0.485	0.144			0.501	0.191	0.506	0.151			0.529	0.205
	CP	0.948	0.802			0.952	0.948	0.944	0.844			0.948	0.944
1.5	Bias	0.013	−0.146	0.031	0.011	0.038	0.051	0.008	−0.126	0.046	0.177	0.038	0.082
	SD	0.597	0.175	0.535	0.198	0.615	0.230	0.584	0.171	0.672	0.323	0.608	0.229
	MSE	0.357	0.052	0.287	0.039	0.380	0.056	0.341	0.045	0.454	0.136	0.371	0.059
	ESE	0.593	0.172			0.601	0.228	0.575	0.171			0.596	0.228
	CP	0.954	0.838			0.956	0.956	0.956	0.864			0.956	0.944
2	Bias	0.021	−0.142	0.020	0.020	0.041	0.051	0.011	−0.131	0.061	0.121	0.035	0.072
	SD	0.711	0.205	0.607	0.224	0.727	0.265	0.650	0.182	0.753	0.351	0.674	0.241
	MSE	0.506	0.062	0.369	0.051	0.530	0.073	0.423	0.050	0.571	0.138	0.456	0.063
	ESE	0.705	0.202			0.718	0.261	0.643	0.190			0.663	0.251
	CP	0.954	0.882			0.952	0.956	0.956	0.890			0.958	0.964

Table 2: Results of the simulation study where $\log(T) = -Z - X + \epsilon$. The number of replications is 500. NV, CW, and SP stand for the naive, Cheng and Wang's method, and the proposed semiparametric approach. Here SD, MSE, ESE, and CP denote the standard deviation of the estimates, mean squared error, estimated standard error based on the formula, and 95% coverage probability. The sample size was $n = 200$ and $U^* \sim \sigma_U \text{Uniform}(-1.75, 1.75)$ with $\sigma_U = 0.71$.

		10% censoring						50% censoring					
		NV		CW		SP		NV		CW		SP	
r		β_1	β_2	β_1	β_2	β_1	β_1	β_2	β_2	β_1	β_2	β_1	β_2
0	Bias	−0.065	−0.199	0.013	0.026	0.045	0.072	−0.039	−0.152	0.065	0.394	0.041	0.114
	SD	0.275	0.087	0.357	0.137	0.313	0.151	0.357	0.104	0.494	0.283	0.391	0.169
	MSE	0.080	0.047	0.128	0.019	0.100	0.028	0.129	0.034	0.248	0.235	0.155	0.042
	ESE	0.266	0.084			0.321	0.170	0.342	0.095			0.386	0.162
	CP	0.926	0.366			0.968	0.978	0.936	0.606			0.946	0.946
0.5	Bias	−0.020	−0.164	0.016	0.032	0.030	0.067	−0.014	−0.132	0.065	0.303	0.033	0.108
	SD	0.377	0.113	0.391	0.156	0.401	0.169	0.445	0.127	0.528	0.273	0.472	0.188
	MSE	0.143	0.040	0.153	0.025	0.162	0.033	0.198	0.034	0.283	0.166	0.224	0.047
	ESE	0.375	0.116			0.399	0.164	0.438	0.126			0.470	0.183
	CP	0.932	0.68			0.936	0.954	0.952	0.788			0.948	0.960
1	Bias	0.002	−0.152	0.023	0.031	0.032	0.059	0.006	−0.131	0.076	0.237	0.043	0.095
	SD	0.487	0.146	0.460	0.172	0.505	0.197	0.509	0.146	0.611	0.296	0.533	0.207
	MSE	0.237	0.044	0.212	0.031	0.256	0.042	0.259	0.038	0.379	0.144	0.286	0.052
	ESE	0.485	0.144			0.501	0.197	0.506	0.149			0.529	0.205
	CP	0.936	0.812			0.936	0.946	0.950	0.838			0.944	0.962
1.5	Bias	0.012	−0.146	0.030	0.018	0.032	0.056	0.008	−0.132	0.060	0.176	0.040	0.083
	SD	0.596	0.172	0.534	0.199	0.610	0.229	0.580	0.169	0.670	0.324	0.594	0.232
	MSE	0.355	0.051	0.286	0.040	0.373	0.056	0.336	0.046	0.453	0.136	0.354	0.061
	ESE	0.593	0.172			0.607	0.227	0.575	0.170			0.596	0.228
	CP	0.946	0.864			0.952	0.950	0.952	0.856			0.954	0.954
2	Bias	0.020	−0.141	0.005	0.043	0.035	0.055	0.011	−0.134	0.098	0.138	0.037	0.075
	SD	0.707	0.202	0.606	0.215	0.719	0.262	0.641	0.184	0.570	0.265	0.661	0.247
	MSE	0.500	0.061	0.367	0.048	0.518	0.072	0.411	0.052	0.335	0.089	0.438	0.067
	ESE	0.706	0.201			0.718	0.261	0.643	0.189			0.663	0.251
	CP	0.946	0.898			0.95	0.948	0.958	0.88			0.954	0.958

W-A5 Files for computation

The software consists of three files: `readme.txt`, `compute_code.txt`, and `simu_subrout.f90` that contains all subroutines. First readme.txt

```
# This code allows you to estimate the parameters of the linear
# transformation model when a covariate is measured with errors.
# Reference: "Semiparametric analysis of linear transformation models
# with covariate measurement errors" by Sinha and Ma.
# Note that the following code can handle the scenario where e has a hazard
# function \lambda_e(t)=exp(t)/\{1+rexp(t)\}.
# Here is an example of estimating the parameters.
# First we simulate a dataset

set.seed(1)
r=0
m=3
trbeta1=1
trbeta2=1
sigmau=0.71
n=200 #Sample size
ran=runif(n, 0, 1)
nby3=as.integer(0.33*n)
x=c(rnorm(nby3, -0.6, 0.5), rnorm((n-nby3), 1.25, 0.5)) # true covariate x
x=x+6
z=runif(n, 0, 1)  # error free covariate
if(r>0) epsilon=log(( (1+0)*exp(-r*log(ran))   -1)/r) else epsilon=log(-log(ran))
t=exp(-trbeta1*z-trbeta2*x+epsilon) # this is the  actual time to event
cns=runif(n, 0, 6) # 10% censored data with r=0
delta=as.numeric(t<=cns) # right censoring indicator
tstar=apply(cbind(t, cns), 1, min)
ustar=matrix(rnorm((m*n), 0, sigmau), ncol=m, nrow=n) # measurement error
wstar=x+ustar
wstar=wstar/sd(wstar)  # It is aways better to use rescaled values
z=as.matrix(z)
## End of data generation
## you need to type in the following code in the R terminal
source("compute_code.txt")
## Call this function
out=ltm.me(tstar, delta, wstar, z, r)
## Output
```

```
## Estimates of the finite dimensional parameters
# due to the naive approach: out$naive.estimate
## Estimated standard errors: out$naive.se
## Estimated variance covariance matrix: out$naive.vcov
out
$naive.estimate
[1] 1.245104 1.196378
$naive.vcov
            [,1]        [,2]
[1,] 0.071383876 0.006531428
[2,] 0.006531428 0.011547962
$naive.se
[1] 0.2671776 0.1074614
$our.estimate
[1] 1.435683 1.738297
$our.vcov
          [,1]       [,2]
[1,] 0.13236273 0.04662215
[2,] 0.04662215 0.08524824
$our.se
[1] 0.3638169 0.2919730
################
################
## Estimates of the finite dimensional parameters
# due to the proposed approach: out$our.estimate
## Estimated standard errors: out$our.se
## Estimated variance covariance matrix: out$our.vcov
##
### In this example we consider two components of Z
set.seed(1)
r=0
m=3
trbeta1=1
trbeta2=1
sigmau=0.21
n=200 #Sample size
ran=runif(n, 0, 1)
nby3=as.integer(0.33*n)
x=c(rnorm(nby3, -0.6, 0.5), rnorm((n-nby3), 1.25, 0.5)) # true covariate x
```

```
z1=runif(n, 0, 1)  # error free covariate
z2=rbinom(n, 1, 0.4)
z=cbind(z1, z2)
if(r>0) epsilon=log(( (1+0)*exp(-r*log(ran))   -1)/r) else epsilon=log(-log(ran))
t=exp(-trbeta1*z1-0.5*z2-trbeta2*x+epsilon) # this is the  actual time to event
cns=runif(n, 0, 6) # 10% censored data with r=0
delta=as.numeric(t<=cns) # right censoring indicator
tstar=apply(cbind(t, cns), 1, min)
ustar=matrix(rnorm((m*n), 0, sigmau), ncol=m, nrow=n) # measurement error
wstar=x+ustar
z=as.matrix(z)
##
out=ltm.me(tstar, delta, wstar, z, r)
> out
$naive.estimate
[1] 1.5182509 0.6556651 1.1446867
$naive.vcov
            [,1]        [,2]        [,3]
[1,] 0.075874695 0.003655317 0.007200891
[2,] 0.003655317 0.024254517 0.003663028
[3,] 0.007200891 0.003663028 0.010144977
$naive.se
[1] 0.2754536 0.1557386 0.1007223
$our.estimate
[1] 1.5387798 0.6433537 1.1800398
$our.vcov
            [,1]        [,2]        [,3]
[1,] 0.079923566 0.003852630 0.008579131
[2,] 0.003852630 0.025416045 0.003901769
[3,] 0.008579131 0.003901769 0.011995941
$our.se
[1] 0.2827076 0.1594241 0.1095260
```

This is `compute_code.txt`

```
ltm.me=function(tstar, delta, wstar, z, r){
#
#
require(statmod)
dyn.load("simu_subrout.so")
#
#

n=length(tstar)
if(length(delta)!=n) stop('Dimensions do not match')
if(nrow(wstar)!=n) stop('Dimensions do not match')
if(nrow(z)!=n) stop('Dimensions do not match')
wstar=wstar-mean(wstar) # recentering
p=ncol(z)
m=ncol(wstar)
##### ordering the data
out=sort(tstar, index.return=T)
tstar=tstar[out$ix]
delta=delta[out$ix]
z=z[out$ix, ]
wstar=wstar[out$ix, ]
wbar=apply(wstar, 1, mean)
nofail=sum(delta) # number of failures
failtime=tstar[delta==1] # the sorted failure times

##### Naive method is use of Chen et al's method where we use wbar instead of x
#
z=as.matrix(z)
storage.mode(z)<-"double"
index=(1:n)[delta==1]
pplus1=p+1
pplus2=p+2
maxcount=1000
tol=0.001
naivebeta=as.double(rep(0, pplus1))
# For the naive estimates
capht=as.double(rep(0, nofail))
```

```
out100=.Fortran("solution1", output=naivebeta, output2=capht,
 delta=as.double(delta), failtime=as.double(failtime),
index=as.integer(index), maxcount=as.integer(maxcount), n=as.integer(n),
 nofail=as.integer(nofail), p=as.integer(p), pplus1=as.integer(pplus1),
 r=as.double(r), tol=as.double(tol), tstar=as.double(tstar),
 wbar=as.double(wbar), z)
naive.estimate=out100$output
#
# Standard error calculation of the Chen's method
cap_sigma_down_star=matrix(0, ncol=(pplus1), nrow=(pplus1))
cap_sigma_up_star=matrix(0, ncol=(pplus1), nrow=(pplus1))
storage.mode(cap_sigma_down_star)<-"double"
storage.mode(cap_sigma_up_star)<-"double"
#
out200=.Fortran("stdforsol1", beta=as.double(out100$output),
 capht=as.double(out100$output2), output1=cap_sigma_down_star,
 output2=cap_sigma_up_star, as.double(delta), as.double(failtime),
 n=as.integer(n), as.double(wbar),  as.integer(nofail), p=as.integer(p),
 pplus1=as.integer(pplus1), as.double(r), as.double(tstar), z)
#
naive.vcov=solve(out200$output1)%*%(out200$output2)%*%t(solve(out200$output1))/n
naive.se=sqrt(diag(naive.vcov))
#
#### Proposed approach
#### The following four lines for hermite quadrature
nnodes=30
out=gauss.quad(nnodes,kind="hermite",alpha=0,beta=0)
xnodes=out$nodes
tnodes=xnodes
wnodes=xnodes
weight=exp(xnodes^2)*out$weights
#### This is our V
eta=as.integer(m/2)
if(eta==1) term1=wstar[, 1]/(2*eta) else term1=apply(wstar[, 1:eta],
 1, sum)/(2*eta)
if((m-eta)==1) term2=wstar[, m]/(2*m-2*eta) else term2=apply(wstar[,
 (eta+1):m], 1, sum)/(2*m-2*eta)
v=term1-term2
#### this is our W
```

```
neww=term1+term2
#### bandwidth for estimation of f_U
h= bw.nrd(v)#width.SJ(v, method = "dpi")
density=function(val) sum(dnorm((v-val)/h))/(n*h)
zmat=cbind(1, z)
pplus2=p+2
newmat=matrix(0, nrow=n, ncol=nnodes)
for( i in 1: n)newmat[i, ]=apply(as.matrix(neww[i]-xnodes), 1, density)

#### Likelihood function of W, f(W|Z, \theta)=\int f(X|Z, \theta)f_U(W-X)dX
indloglk=function(para){
lk=rep(0, n)
for( i in 1: n){
tempo=sum(weight*newmat[i, ]*exp( -0.5*(xnodes-zmat[i, ]%*%para[1:pplus1])^2/
para[pplus2])/sqrt(para[pplus2]))
tempo=max(tempo, 1e-300)
lk[i]=log(tempo)}
return(lk)
}
loglk=function(para)-sum(indloglk(para))


###### The following lines determines the initial parameter values for \theta
outold=lm(neww~z)
upperl=c(outold$coef+3*sqrt(diag(summary(outold)$cov.unscaled)), var(wbar))
lowerl=c(outold$coef-3*sqrt(diag(summary(outold)$cov.unscaled)),
 (var(wbar)-0.25*max(apply(wstar, 1, var))/m))
##### Estimation of $\theta$ by maximizing f(W|Z, \theta)
out=optim(c(outold$coef, 0.5*var(wbar)), loglk, method="L-BFGS-B",
 lower=lowerl, upper=upperl, hessian=T)
##### A_{W|Z}
cap.a.w.given.z=-solve(out$hessian/n)


theta=out$par
gamma=out$par[1:pplus1]
sigma2x=out$par[pplus2]
###### f(X|Z, theta),
fxgivenz=function(x, z0, theta)
```

```
{
gamma=out$par[1:pplus1]
sigma2x=out$par[pplus2]
den=rep(0, ncol=length(x))
den=exp(-0.5*(x-c(1, z0)%*%gamma)^2/sigma2x)/sqrt(2*pi*sigma2x)
list(density=den)
}
###### f(X|W, Z)
jointdensity=matrix(0, nrow=n, ncol=nnodes)
for( i in 1:n){
jointdensity[i, ]= fxgivenz(xnodes, z[i, ], theta)$density*newmat[i, ]*weight
tempo.sum=sum(jointdensity[i, ])
if(tempo.sum!=0) jointdensity[i, ]=jointdensity[i, ]/sum(jointdensity[i, ])
}
fxgivenwnz=jointdensity
###### Initialization of some parameters
ourbeta=as.double(rep(0, pplus1))
storage.mode(fxgivenwnz)<-"double"
caph=as.double(rep(0, n))
capht=as.double(rep(0, nofail))
###### Estimation of beta
###### Untill the standard error calculation is fixed, we turn off
###### the following 5 lines.
 newout=.Fortran("solution2", output1=ourbeta, output2=caph, output3=capht,
 delta=as.double(delta), fxgivenwnz, failtime=as.double(failtime),
 index=as.integer(index), maxcount=as.integer(maxcount),
 n=as.integer(n), nnodes=as.integer(nnodes), nofail=as.integer(nofail),
 p=as.integer(p), pplus1=as.integer(pplus1), r=as.double(r), tol=as.double(tol),
 tstar=as.double(tstar), wbar=as.double(neww), xnodes=as.double(xnodes), z)
  # alphabetical order
###### Storing our estimates
our.estimate=newout$output1
#
# Standard error calculation
###
###
storage.mode(cap.a.w.given.z)<-"double"
cap_sigma_1=matrix(0, ncol=pplus1, nrow=pplus1)
sigma_star=matrix(0, ncol=pplus1, nrow=pplus1)
```

```
storage.mode(cap_sigma_1)<-"double"
storage.mode(sigma_star)<-"double"
######################
storeden=NULL;
den=as.double(rep(0, nnodes))
for( i in 1:n){
outden=.Fortran("densityofxvecgivenwnz",
output=den, h=as.double(h),  n=as.integer(n), nnodes=as.integer(nnodes),
 ntheta=as.integer(length(theta)), p=as.integer(p), theta=as.double(theta),
 v=as.double(v), as.double(neww[i]), as.double(weight), as.double(xnodes),
 as.double(z[i, ]))
storeden=rbind(storeden, outden$output)
}
storage.mode(storeden)<-"double"
storewz=NULL;
storexz=NULL;
lpipel=NULL;
lpipeu=NULL;
#for( l in 4: (nofail-3)){
for(l in 1:nofail){
###
capy=rep(0, n)
capy[tstar>=failtime[l]]<-1;
if(mean(capy)>0.975) {lpipel=c(lpipel, l)} else {
if(mean(capy)<0.025){lpipeu=c(lpipeu, l)} else {
outwz=glm(capy~z+neww, family=binomial)
storewz=rbind(storewz, outwz$coef)

###
 lglkfnc=function(gamma){
lglk=as.double(0)
outneglk= .Fortran("neglkfunc",as.double(capy), as.double(gamma), output=lglk,
n=as.integer(n), nnodes=as.integer(nnodes),  p=as.integer(p),
pplus2=as.integer(pplus2), storeden, as.double(xnodes),z)
return(outneglk$output)
}
lowerl= as.numeric(summary(outwz)$coef[, 1]-2* summary(outwz)$coef[, 2])
upperl= as.numeric(summary(outwz)$coef[, 1]+2* summary(outwz)$coef[, 2])
if(is.nan(lglkfnc(upperl)))  upperl=as.numeric(summary(outwz)$coef[, 1]+
```

```
1*summary(outwz)$coef[, 2])
if(is.nan(lglkfnc(lowerl)))  lowerl=as.numeric(summary(outwz)$coef[, 1]-
1*summary(outwz)$coef[, 2])
outxz=optim(rep(0.5, (pplus2)), lglkfnc, method="L-BFGS-B", lower=lowerl,
 upper=upperl)
storexz=rbind(storexz, outxz$par)
}
}
}
######################
storage.mode(storexz)<-"double"
storage.mode(storewz)<-"double"
######################
stdcal=.Fortran("stdforsol2", beta=as.double(newout$output1),
capht=as.double(newout$output3), output1=cap_sigma_1,
delta=as.double(delta), failtime=as.double(failtime), h=as.double(h),
cap.a.w.given.z, ll=as.integer(length(lpipel)), lu= as.integer(length(lpipeu)),
 n=as.integer(n), neww=as.double(neww), nnodes=as.integer(nnodes),
 nofail=as.integer(nofail), ns=as.integer(nofail-length(lpipel)- length(lpipeu)),
 ntheta=as.integer(length(theta)), p=as.integer(p), pplus1=as.integer(pplus1),
 pplus2=as.integer(pplus2),  r=as.double(r), sd_neww=as.double(sd(neww)),
 sd_z=as.double(apply(z, 2, sd)), output2=sigma_star, storewz, storexz,
theta=as.double(theta), tstar=as.double(tstar), v=as.double(v),
 weight=as.double(weight), xnodes=as.double(xnodes), z)

our.vcov=solve(stdcal$output1)%*%(stdcal$output2)%*%t(solve(stdcal$output1))/n
#source("std_data.R")
our.se= sqrt(diag(our.vcov))
#
result<-list(naive.estimate, naive.vcov, naive.se, our.estimate,
our.vcov, our.se)
names(result)<-c("naive.estimate", "naive.vcov", "naive.se",
  "our.estimate", "our.vcov", "our.se")
return(result)
}
```

This is simu_subrout.f90.

```
 subroutine  solution1(beta, capht, delta, failtime, &
 index, maxcount, n, nofail,  &
 p, pplus1, r, tol, tstar, wbar, z)  ! alphabetical order
!
 implicit none
 ! input output variables
 integer :: maxcount, n, nofail, p, pplus1
 real*8 :: beta(pplus1), capht(nofail), delta(n), failtime(nofail), &
   r, tol, tstar(n),  wbar(n), z(n, p)
 integer :: index(nofail)
! local variables
 integer ::  count, k, i1, newcount
 real*8 :: beta1(p), beta2, caph(n), ee, eed, eedmat(pplus1, pplus1),&
 eest(pplus1), eps, eta(n),  h(pplus1), neweps, newt(n), newtd(n),&
 newz(n, p), oldbeta(pplus1), para, tempo(n), tempo1(n), tempo2(n), zeta(n)

 do i1=1, n
 caph(i1)=-10000000.d0
 end do
 do i1=1, nofail
  capht(i1)=0.d0
 end do
 neweps=2.d0
 newcount=0
 beta=  0*(/(i1, i1=1,(pplus1), 1)/)+0.25d0

 if(r.eq.0.d0) then
  do while((neweps.gt.tol) .and. (newcount.lt.maxcount))
   newcount=newcount+1
   beta1=beta(1:p)
   beta2=beta(p+1)
   eta=matmul(z, beta1)+wbar*beta2
   count=0
   eps=2
   para=0.01d0
   do while ((eps.gt.tol) .and. (count.lt.maxcount))
    count=count+1
    zeta=exp(eta+para)
```

```
    newt=zeta
    ee=sum(newt(index(1): n))-1
    newtd=zeta
    eed=sum(newtd(index(1): n))
    para=para-ee/eed
    eps=abs(ee/eed)
   end do
 !  print*, 'eps= ', eps, ' para=', para
   capht(1)=para
   where (tstar.ge.failtime(1)) caph=capht(1)
!
   do k=2, nofail
    tempo1=exp(eta+capht(k-1))
    count=0
    eps=2.d0
    para=capht(k-1)+0.001d0
    do while ((eps.gt.tol) .and. (count.lt.maxcount))
     count=count+1
     tempo2=exp(eta+para)
     newt=tempo2-tempo1
     ee=sum(newt(index(k): n))-1
     newtd=tempo2
     eed=sum(newtd(index(k): n))
     para=para-ee/eed
     eps=abs(ee/eed)
    end do
    capht(k)=para
! print*,'k= ', k, ' eps= ', eps,  ' para= ', para
    where (tstar.ge.failtime(k)) caph=capht(k)
   end do
!
   oldbeta=beta
   count=0
   eps=2.d0
   do while ((eps.gt.tol) .and. (count.lt.maxcount))
    count=count+1
    beta1=beta(1:p)
    beta2=beta(pplus1)
    eest= 0*(/(i1, i1=1,(pplus1), 1)/)
```

```
   eta=matmul(z, beta1)+wbar*beta2+caph
   tempo=delta-exp(eta)
   eest(1:p)=matmul(transpose(z), tempo)
   eest(pplus1)=dot_product(tempo, wbar)
   do i1=1, p
    newz(:, i1)=z(:, i1)*exp(eta)
   end do
   eedmat(1:p, 1:p)= - matmul(transpose(z), newz)
   eedmat(pplus1, 1:p)=-matmul(wbar, newz)
   eedmat(1:p, pplus1)=eedmat(pplus1, 1:p)
   eedmat(pplus1, pplus1)=-sum(wbar**2*exp(eta))
   call gaussj(eedmat,pplus1, pplus1)
   h=matmul(eedmat, eest)
   beta=beta-h
   eps=sum(abs(h/oldbeta))
  end do
  neweps=sum(abs((oldbeta-beta)/beta))
 end do
else
 do while((neweps.gt.tol) .and. (newcount.lt.maxcount))
  newcount=newcount+1
  beta1=beta(1:p)
  beta2=beta(p+1)
  eta=matmul(z, beta1)+wbar*beta2
  count=0
  eps=2
  para=0.01d0
  do while ((eps.gt.tol) .and. (count.lt.maxcount))
   count=count+1
   zeta=exp(eta+para)
   newt=log(1.d0+r*zeta)
   ee=sum(newt(index(1): n))-r
   newtd=r/(r+(1.d0/zeta))
   eed=sum(newtd(index(1): n))
   para=para-ee/eed
   eps=abs(ee/eed)
  end do
 ! print*, 'eps= ', eps, ' para=', para
  capht(1)=para
```

```
   where (tstar.ge.failtime(1)) caph=capht(1)
!
   do k=2, nofail
    tempo1=log(1+r*exp(eta+capht(k-1)))
    count=0
    eps=2.d0
    para=capht(k-1)+0.001d0
    do while ((eps.gt.tol) .and. (count.lt.maxcount))
     count=count+1
     tempo2=exp(eta+para)
     newt=log(1+r*tempo2)-tempo1
     ee=sum(newt(index(k): n))-r
     newtd=r/(r+(1.d0/tempo2))
     eed=sum(newtd(index(k): n))
     para=para-ee/eed
     eps=abs(ee/eed)
    end do
    capht(k)=para
    where (tstar.ge.failtime(k)) caph=capht(k)
   end do
!
   oldbeta=beta
   count=0
   eps=2.d0
   do while ((eps.gt.tol) .and. (count.lt.maxcount))
    count=count+1
    beta1=beta(1:p)
    beta2=beta(pplus1)
    eest= 0*(/(i1, i1=1,(pplus1), 1)/)
    eta=matmul(z, beta1)+wbar*beta2+caph
    tempo=delta-(1.d0/r)*log(1.d0+r*exp(eta))
    eest(1:p)=matmul(transpose(z), tempo)
    eest(pplus1)=dot_product(tempo, wbar)
    do i1=1, p
     newz(:, i1)=z(:, i1)/(exp(-eta)+r)
    end do
    eedmat(1:p, 1:p)= - matmul(transpose(z), newz)
    eedmat(pplus1, 1:p)=-matmul(wbar, newz)
    eedmat(1:p, pplus1)=eedmat(pplus1, 1:p)
```

```
    eedmat(pplus1, pplus1)=-sum(wbar**2/(exp(-eta)+r))
    call gaussj(eedmat,pplus1, pplus1)
    h=matmul(eedmat, eest)
    beta=beta-h
    eps=sum(abs(h/oldbeta))
   end do
   neweps=sum(abs((oldbeta-beta)/beta))
  end do
 endif
 return
 end  subroutine
!
! this is the standard error calculation for Chen's method
 subroutine  stdforsol1(beta, capht, cap_sigma_down_star,  &
 cap_sigma_up_star, delta, failtime, &
 n, wbar,  nofail, p, pplus1, r, tstar, z)  ! alphabetical order
 implicit none
 integer :: n,  nofail,  p, pplus1
 real*8 :: beta(pplus1), capht(nofail), cap_sigma_down_star(pplus1, &
  pplus1), cap_sigma_up_star(pplus1, pplus1), delta(n), &
  failtime(nofail), wbar(n), r, tstar(n), z(n, p)
!
! local variables
  integer:: i, it, i1,  j, j1, j2
!
  real*8:: cap_b, cov_mat(n, pplus1), cov_mat_bar_t(nofail, pplus1), &
  capy(n, nofail), dcapht(nofail), eta(n), lambda(n, nofail), &
  deriv_lambda(n, nofail), temp_eta(nofail), &
  tempo(nofail), tempo1, tempo2(nofail), tempo3(nofail, pplus1)
!
! New covariate matrix
  cov_mat(:, 1:p)=z
  cov_mat(:, pplus1)=wbar
! Estimation  of dH_0(t)
 dcapht(1)=0.d0!abs(capht(1))!0.d0!capht(1)
 dcapht(2:nofail)=capht(2:nofail)-capht(1:(nofail-1))
!
  do i=1, n
   capy(i, :)=0*(/(i1, i1=1,(nofail), 1)/)
```

```
   where(failtime.le. tstar(i)) capy(i, :)=1.d0
  end do
!print*, 'hihi'
    eta=matmul(cov_mat, beta)
!   print*, 'eta = ',eta(1:5)
  if(r.eq.0.d0) then
   do i=1, n
    temp_eta=eta(i)+capht
    lambda(i, :)=exp(temp_eta)
    do it=1, nofail
     deriv_lambda(i, it)= lambda(i, it)
    end do
   end do
  else
   do i=1, n
    temp_eta=eta(i)+capht
    lambda(i, :)=1.d0/(r+exp(-temp_eta))
    do it=1, nofail
     deriv_lambda(i, it)= lambda(i, it)/(1.d0+r*exp(temp_eta(it)))
    end do
   end do
  endif
  do it=1, nofail
    tempo(it)=sum(deriv_lambda(:, it)*capy(:, it))/sum(lambda(:,&
    it)*capy(:, it))
  end do
!  print*, 'hohoh'
  tempo2=0.d0
  tempo3=0.d0
   do it=1, nofail
    do i=1, n
     if(tstar(i).le. failtime(it)) then
      tempo1=sum(tempo(1:it)* dcapht(1:it)*(1.d0-capy(i, 1:it))  )

      if(tstar(i).eq.failtime(it)) then
        cap_b=exp(tempo1+tempo(it)* dcapht(it))
      else
        cap_b=exp(tempo1)
      endif
```

```
     else
      cap_b=1.d0
     endif
     tempo2(it)=tempo2(it)+lambda(i, it)*capy(i, it)
     do j=1, pplus1
      tempo3(it, j)=tempo3(it, j)+cap_b*cov_mat(i, j)* &
      lambda(i, it)*capy(i, it)
     end do
    end do
   end do
!  print*, 'hehe'
!  print*, lambda(1, :)
   do it=1, nofail
    do j=1, pplus1
     cov_mat_bar_t(it, j)=tempo3(it, j)/tempo2(it)
    end do
   end do
! print*, 'hahaha'
!
   do j1=1, pplus1
    do j2=1, pplus1
     cap_sigma_up_star(j1, j2)= 0.d0
     cap_sigma_down_star(j1, j2)= 0.d0
     do it=1, nofail
      cap_sigma_up_star(j1, j2)=cap_sigma_up_star(j1, j2)+&
      sum((cov_mat(:, j1)-cov_mat_bar_t(it, j1))*(  &
      cov_mat(:, j2)-cov_mat_bar_t(it, j2))*lambda(:, it)*&
      capy(:, it)*dcapht(it))
!
      cap_sigma_down_star(j1, j2)=cap_sigma_down_star(j1, j2)+&
      sum((cov_mat(:, j1)-cov_mat_bar_t(it, j1))*  &
      (cov_mat(:, j2)-cov_mat_bar_t(it, j2))*deriv_lambda(:, it)*&
      capy(:, it)*dcapht(it))
     end do
     cap_sigma_up_star(j1, j2)=cap_sigma_up_star(j1, j2)/float(n)
     cap_sigma_down_star(j1, j2)=cap_sigma_down_star(j1, j2)/float(n)
    end do
   end do
  return
```

```
  end subroutine
!
!
! This is our method
 subroutine  solution2(beta, caph, capht, delta, densityofxgivenwnz,&
  failtime, index, maxcount, n, nnodes, nofail,  &
 p, pplus1, r, tol, tstar, wbar, xnodes, z)  ! alphabetical order
!
 implicit none
 ! input output variables
 integer :: maxcount, n, nnodes, nofail, p, pplus1
 real*8 ::beta(pplus1), caph(n), capht(nofail), delta(n), &
  densityofxgivenwnz(n, nnodes), failtime(nofail), r, tol, tstar(n),&
   wbar(n), xnodes(nnodes), z(n, p)
 integer :: index(nofail)
! local variables
 integer ::  count, k, i, i1, newcount

 real*8 :: beta1(p), beta2, beta_hold(pplus1),  caplambda(n), &
  caplambda_1(n), caplambda_k(n), caplambda_kminus(n),&
  deriv_caplambda_1(n), deriv_caplambda_k(n),  &
  deriv_qnty0_multx(nnodes), ee, eed, eedmat(pplus1, pplus1),&
  eest(pplus1), eps, eta(n),  h(pplus1), neweps, neweta(nnodes), &
  newt(n), oldbeta(pplus1), para, qnty0(nnodes), &
  deriv_qnty0(nnodes),  qntyz(n, p), qntyx(n), tempo1(n),&
  tempo2(n), tempo3(n), maxtstar
  maxtstar=maxval(tstar)
  print*, 'hello hello'
 do i1=1, n
  caph(i1)=-10000000.d0
 end do
 do i1=1, nofail
  capht(i1)=0.d0
 end do

 neweps=2.d0
 newcount=0
 beta=  0*(/(i1, i1=1,(pplus1), 1)/)+0.05d0
 if(r==0.d0) then
```

```
! --------------
  do while((neweps.gt.tol) .and. (newcount.lt.maxcount))
   newcount=newcount+1
   beta1=beta(1:p)
   beta2=beta(p+1)
   eta=matmul(z, beta1)
   count=0
   eps=2
   para=0.01d0
   do while ((eps.gt.tol) .and. (count.lt.maxcount))
    count=count+1
    do i=1, n
     neweta=eta(i)+xnodes*beta2+para
     qnty0=exp ( -exp(neweta))
     deriv_qnty0=-qnty0*exp(neweta)
     tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
     tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
    end do
   caplambda_1=-log(tempo1)
   deriv_caplambda_1= -tempo2/tempo1
   ee=sum(caplambda_1(index(1): n))-1
   eed=sum(deriv_caplambda_1(index(1): n))
   para=para-ee/eed
   eps=abs(ee/eed)
   end do
!  print*, 'for capht(1)  ',' eps= ', eps, ' para=', para
   capht(1)=para
   where (tstar.ge.failtime(1)) caph=capht(1)
   if(failtime(nofail)==maxval(tstar)) then

    do k=2, (nofail-1)
     do i=1, n
      neweta=eta(i)+xnodes*beta2+capht(k-1)
      qnty0=exp ( -exp(neweta))
      tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
     end do
     caplambda_kminus=-log(tempo1)
     count=0
     eps=2.d0
```

```
  para=capht(k-1)+0.1d0
  do while ((eps.gt.tol) .and. (count.lt.maxcount))
   count=count+1
   do i=1, n
    neweta=eta(i)+xnodes*beta2+para
    qnty0=exp ( -exp(neweta))
    deriv_qnty0=-qnty0*exp(neweta)
    tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
    tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
   end do
   caplambda_k= -log(tempo1)
   deriv_caplambda_k=-tempo2/tempo1
   newt=caplambda_k-caplambda_kminus
   ee=sum(newt(index(k): n))-1
   eed=sum(deriv_caplambda_k(index(k): n))
   para=para-ee/eed
   eps=abs(ee/eed)
  end do
  capht(k)=para
  where (tstar.ge.failtime(k)) caph=capht(k)
 end do
 capht(nofail)=capht(nofail-1)
else
 do k=2, nofail
  do i=1, n
   neweta=eta(i)+xnodes*beta2+capht(k-1)
   qnty0=exp ( -exp(neweta))
   tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
  end do
  caplambda_kminus=-log(tempo1)
  count=0
  eps=2.d0
  para=capht(k-1)+0.1d0
  do while ((eps.gt.tol) .and. (count.lt.maxcount))
   count=count+1
   do i=1, n
    neweta=eta(i)+xnodes*beta2+para
    qnty0=exp ( -exp(neweta))
    deriv_qnty0=-qnty0*exp(neweta)
```

```
      tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
      tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
     end do
     caplambda_k= -log(tempo1)
     deriv_caplambda_k=-tempo2/tempo1
     newt=caplambda_k-caplambda_kminus
     ee=sum(newt(index(k): n))-1
     eed=sum(deriv_caplambda_k(index(k): n))
     para=para-ee/eed
     eps=abs(ee/eed)
    end do
    capht(k)=para
    where (tstar.ge.failtime(k)) caph=capht(k)
   end do
  endif
!
  oldbeta=beta
  count=0
  eps=2.d0
  do while ((eps.gt.tol) .and. (count.lt.maxcount))
   count=count+1
   beta1=beta(1:p)
   beta2=beta(p+1)
   eest= 0*(/(i1, i1=1,(pplus1), 1)/)
   do i=1, n
    neweta=dot_product(z(i,:), beta1)+xnodes*beta2+caph(i)
    qnty0=exp ( -exp(neweta))
    deriv_qnty0=-qnty0*exp(neweta)
    deriv_qnty0_multx=deriv_qnty0*xnodes
    tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
    tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
    tempo3(i)=dot_product(deriv_qnty0_multx, densityofxgivenwnz(i, :))
    qntyz(i,: )=z(i, :)*tempo2(i)/tempo1(i)
    qntyx(i)=tempo3(i)/tempo1(i)
   end do
   caplambda=-log(tempo1)
   eest(1:p)=matmul(transpose(z), (delta-caplambda) )
   eest(pplus1)=dot_product(wbar, (delta-caplambda) )
   eedmat(1:p, 1:p)=  matmul(transpose(z), qntyz)
```

```
    eedmat(pplus1, 1:p)=matmul(wbar, qntyz)
    eedmat(1:p, pplus1)=matmul(transpose(z), qntyx)
    eedmat(pplus1, pplus1)=sum(wbar*qntyx)
    call gaussj(eedmat,pplus1, pplus1)
    beta_hold=beta
    h=matmul(eedmat, eest)
    beta=beta-h
    eps=sum(abs(h/beta_hold))
   end do
   neweps=sum(abs((oldbeta-beta)/oldbeta))
  end do
! ----------
 else
  do while((neweps.gt.tol) .and. (newcount.lt.maxcount))
   newcount=newcount+1
   beta1=beta(1:p)
   beta2=beta(p+1)
   eta=matmul(z, beta1)
   count=0
   eps=2
   para=0.01d0
   do while ((eps.gt.tol) .and. (count.lt.maxcount))
    count=count+1
    do i=1, n
     neweta=eta(i)+xnodes*beta2+para
     qnty0=exp ( -(1.d0/r)*log(1.d0+r*exp(neweta)))
     deriv_qnty0=-qnty0/(r+exp(-neweta))
     tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
     tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
    end do
    caplambda_1=-log(tempo1)
    deriv_caplambda_1= -tempo2/tempo1
    ee=sum(caplambda_1(index(1): n))-1
    eed=sum(deriv_caplambda_1(index(1): n))
    para=para-ee/eed
    eps=abs(ee/eed)
   end do
 ! print*, 'eps= ', eps, ' para=', para
   capht(1)=para
```

```
where (tstar.ge.failtime(1)) caph=capht(1)
do k=2, nofail
 do i=1, n
  neweta=eta(i)+xnodes*beta2+capht(k-1)
  qnty0=exp ( -(1.d0/r)*log(1.d0+r*exp(neweta)))
  tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
 end do
 caplambda_kminus=-log(tempo1)
 count=0
 eps=2.d0
 para=capht(k-1)+0.001d0
 do while ((eps.gt.tol) .and. (count.lt.maxcount))
  count=count+1
  do i=1, n
   neweta=eta(i)+xnodes*beta2+para
   qnty0=exp ( -(1.d0/r)*log(1.d0+r*exp(neweta)))
   deriv_qnty0=-qnty0/(r+exp(-neweta))
   tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
   tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
  end do
  caplambda_k= -log(tempo1)
  deriv_caplambda_k=-tempo2/tempo1
  newt=caplambda_k-caplambda_kminus
  ee=sum(newt(index(k): n))-1
  eed=sum(deriv_caplambda_k(index(k): n))
  para=para-ee/eed
  eps=abs(ee/eed)
 end do
 capht(k)=para
 where (tstar.ge.failtime(k)) caph=capht(k)
end do
oldbeta=beta
count=0
eps=2.d0
do while ((eps.gt.tol) .and. (count.lt.maxcount))
 count=count+1
 beta1=beta(1:p)
 beta2=beta(p+1)
 eest= 0*(/(i1, i1=1,(pplus1), 1)/)
```

```
    do i=1, n
     neweta=dot_product(z(i,:), beta1)+xnodes*beta2+caph(i)
     qnty0=exp ( -(1.d0/r)*log(1.d0+r*exp(neweta)))
     deriv_qnty0=-qnty0/(r+exp(-neweta))
     deriv_qnty0_multx=deriv_qnty0*xnodes
     tempo1(i)=dot_product(qnty0, densityofxgivenwnz(i, :))
     tempo2(i)=dot_product(deriv_qnty0, densityofxgivenwnz(i, :))
     tempo3(i)=dot_product(deriv_qnty0_multx,&
     densityofxgivenwnz(i, :))
     qntyz(i,: )=z(i, :)*tempo2(i)/tempo1(i)
     qntyx(i)=tempo3(i)/tempo1(i)
    end do
    caplambda=-log(tempo1)
    eest(1:p)=matmul(transpose(z), (delta-caplambda) )
    eest(pplus1)=dot_product(wbar, (delta-caplambda) )
    eedmat(1:p, 1:p)=  matmul(transpose(z), qntyz)
    eedmat(pplus1, 1:p)=matmul(wbar, qntyz)
    eedmat(1:p, pplus1)=matmul(transpose(z), qntyx)
    eedmat(pplus1, pplus1)=sum(wbar*qntyx)
    call gaussj(eedmat,pplus1, pplus1)
    beta_hold=beta
    h=matmul(eedmat, eest)
    beta=beta-h
    eps=sum(abs(h/beta_hold))
   end do
   neweps=sum(abs((oldbeta-beta)/beta))
  end do
 endif
 return
 end  subroutine
!
! this is the standard error calculation for our method
 subroutine  stdforsol2(beta, capht, cap_sigma_1,  delta, &
   failtime, h, inv_a_wnz, ll, lu, n, neww, nnodes, nofail, ns, &
  ntheta, p, pplus1, pplus2, r, sd_neww, sd_z,&
  sigma_star, storewz, storexz, theta, tstar, v, &
  weight, xnodes, z)  ! alphabetical order
 implicit none
 integer :: ll, lu, n, nnodes, nofail,  ns, ntheta, p, pplus1, pplus2
```

```
 real*8 :: beta(pplus1), capht(nofail), delta(n), &
  failtime(nofail), h, inv_a_wnz(ntheta, ntheta), &
  neww(n), r, sd_neww, sd_z(p), storewz(ns, pplus2), &
 storexz(ns, pplus2), theta(ntheta), tstar(n), v(n),  &
  weight(nnodes),  xnodes(nnodes), z(n, p)
! local variables
 integer :: i, i1, it, it1, iw, ix, j,  k, l, l1, l2, mi
 real*8 :: beta1(p), beta2, cap_d_1(nofail), cap_d_1_zstar(nofail,&
  pplus1), cap_d_2(nofail, ntheta), cap_d_2_zstar(nofail, pplus1, &
   ntheta), capg( nnodes, nofail), capj(n, nofail), &
 cap_phi(n, nofail, pplus1),  cap_q_1(n, nofail), &
  cap_q_zstar(n, nofail, pplus1), &
  cap_sigma_1(pplus1, pplus1), cap_sigma_1_1st(pplus1, pplus1), &
 cap_sigma_1_2nd(pplus1, pplus1), cap_sigma_1_3rd(pplus1, pplus1), &
 capy(n, nofail),   &
 cd(nofail), cn(nofail),  dcapht(nofail), den(nnodes), denominator(n,&
  nnodes), densityofxgivenwnz(n, nnodes), denxgivenwnz(nnodes),&
 deriv_capj_wrt_beta(nofail, pplus1), deriv_capj_wrt_caph(n, nofail), &
 eta(n), eta_xnodes(n, nnodes), eta_xnodes_caph(n, nnodes, nofail), &
 e_yt_deriv_capj_wrt_beta(nofail, pplus1), &
 final_cap_d_2(nofail, ntheta),final_cap_d_2_zstar(nofail,pplus1, ntheta),&
 gamma_1(nofail, pplus1), gamma_2(nofail, pplus1), intsw(n, ntheta),&
  lambda_star_caph(nofail), new_capg(nnodes), mean_tempo500, &
 new_capj(n,  nnodes), new_tempo_eta(nnodes), new_tempo_lambda( nnodes),&
  new_tempo_caplambda( nnodes), oldden(n, nnodes, nnodes), &
 orgcaplambda(n, nnodes), orgderiv_lambda(n, nnodes),orglambda(n, nnodes),&
 pi, pr_yt_w_z(n,  nnodes), pr_yt_x_z(n, nnodes), qnty1(nnodes), &
 qnty2(nnodes),  qnty3(nnodes), qnty7(pplus1), qnty8(pplus1), &
 qnty9(n),qnty10(pplus1),qnty11(pplus1), qnty200(n, nnodes),&
 ratio_cn_2_cd(nofail), sigma_star(pplus1, pplus1),&
 sigma_star_1st(pplus1, pplus1), sigma_star_2nd(pplus1, pplus1), sum_temp1,&
 sum_tempo400, swtheta(n, ntheta), store_sxtheta(n, nnodes, ntheta),&
 sxtheta(nnodes,  ntheta), temp1(n), tempo100, tempo200(nnodes),&
 tempo300(n, nnodes), tempo400(nnodes), tempo500(nnodes),&
 tempo600(nofail), tempo700,  tempo800, tempo900(nofail, pplus1),&
 tempo1000(n, pplus1), tempo_mat(pplus1, ntheta), tmp_eta(nnodes),&
 wnodes(nnodes),  zstar(n, pplus1)

! Initialize
```

```
 pi=22.d0/7.d0
 wnodes=xnodes
 do mi=1, 5
 !  print*, mi
 end do
 !print*, 'CHECK====  mi= ', mi
 !
 zstar(:, 1:p)=z
 zstar(:, pplus1)=neww
 !
 cd=0.d0
 cn=0.d0
 !
 !
 cap_sigma_1_1st=0.d0
 cap_d_1=0.d0
 cap_d_1_zstar=0.d0
 cap_d_2=0.d0
 cap_d_2_zstar=0.d0
 cap_q_1= 0.d0
 cap_q_zstar=0.d0
 e_yt_deriv_capj_wrt_beta=0.d00
 tempo900=0.d0
 cap_sigma_1_1st=0.d0
 cap_sigma_1_2nd=0.d0
 cap_sigma_1_3rd=0.d0
 cap_d_1=0.d0
 cap_d_2=0.d0
 cap_d_1_zstar=0.d0
 cap_d_2_zstar=0.d0
 cap_q_1=0.d0
 cap_q_zstar=0.d0
 cap_phi=0.d0
 sigma_star=0.d0
 sigma_star_1st=0.d0
 sigma_star_2nd=0.d0
!
! Estimation  of dH_0(t)
 dcapht(1)=0.d0!abs(capht(1))!0.d0!capht(1)
```

```
 dcapht(2:nofail)=capht(2:nofail)-capht(1:(nofail-1))
!
 do i=1, n
  capy(i, :)=0*(/(i1, i1=1,(nofail), 1)/)
  where(failtime.le. tstar(i)) capy(i, :)=1.d0
 end do
 beta1=beta(1:p)
 beta2=beta(pplus1)
 eta=matmul(z, beta1)
 do i=1, n
  eta_xnodes(i, :)=eta(i)+beta2*xnodes
 end do
!
! calling two subroutines
!
  do j=1, n
   do iw=1, nnodes
    call densityofxvecgivenwnz(den, h,  n, nnodes, ntheta, p, theta, &
      v, wnodes(iw), weight, xnodes, z(j, :))
     oldden(j, iw, :)=den
!     print*, 'j= ', j, ' iw= ', iw, 'den= ', den
    end do
   end do
 do i=1, n
   call sxgivenz( nnodes, ntheta, p, sxtheta, theta, xnodes, z(i, :))
  store_sxtheta(i, :, :)=sxtheta
 end do
 call swgivenz(h, n, neww, nnodes, ntheta, p, swtheta, theta, v, weight,&
  xnodes, z)
 call intswgivenz(h, intsw, n, nnodes, ntheta, p, theta, v, weight, xnodes, z)
!
 do i=1, n
  call densityofxvecgivenwnz(denxgivenwnz, h,  n, nnodes,  ntheta, p, &
   theta, v, neww(i), weight, xnodes, z(i, :))
!
   densityofxgivenwnz(i, :)=denxgivenwnz
 end do
!  ************* Beginning of the main loop for i, the
! index for subject (i.e., i=1, \cdots, n)
```

```
!
   do it=1, nofail
    if(it.le.ll) then
     pr_yt_w_z=1.d0
     pr_yt_x_z=1.d0
    else
     if(it.gt.(nofail-lu)) then
      pr_yt_w_z=0.d0
      pr_yt_x_z=0.d0
     else
      do i=1, n
       tmp_eta= storewz((it-ll), 1)+sum(storewz((it-ll), 2:pplus1)*z(i, 1:p))+&
       storewz((it-ll), pplus2)*xnodes
       pr_yt_w_z(i, :)=1.d0/(1.d0+exp(-tmp_eta))
       tmp_eta= storexz((it-ll), 1)+sum(storexz((it-ll), 2:pplus1)*z(i, 1:p))+&
       storexz((it-ll), pplus2)*xnodes
       pr_yt_x_z(i, :)=1.d0/(1.d0+exp(-tmp_eta))
      end do
     endif
    endif
    tempo300=exp(eta_xnodes+capht(it))
    if(r.eq.0.d0) then
     orglambda=tempo300
     orgcaplambda=tempo300
     orgderiv_lambda=tempo300
    else
     orglambda=1.d0/(r+(1.d0/tempo300) )
     orgcaplambda=(1/r)*log(1.d0+ r*tempo300)
     orgderiv_lambda=orglambda/(1.d0+r*tempo300)
    endif
!!!!! need to fix this part
   do j=1, n
    do iw=1, nnodes
     tempo200=exp(-orgcaplambda(j, :)+log(oldden(j, iw, :)) )
     denominator(j,  iw)=sum(tempo200)
     if(denominator(j,  iw).eq.0.d0) denominator(j,  iw)=10.0**(-10)
     new_capg=  tempo200/denominator(j,  iw)
     new_capj(j,  iw)= sum(orglambda(j, :)*new_capg)
     qnty200(j, iw)=sum(pr_yt_x_z(j,  :)*(orglambda(j, :)- &
```

```
     new_capj(j, iw))*new_capg)
    end do
   end do
   do i=1, n
    capg(:, it)=exp(-orgcaplambda(i, :))*densityofxgivenwnz(i, :)/&
    sum(exp(-orgcaplambda(i, :))*densityofxgivenwnz(i, :))
!
!
   qnty1=exp(-orgcaplambda(i, :))
   qnty2=orglambda(i, :)*qnty1
!
   qnty3=orgderiv_lambda(i, :)-orglambda(i, :)**2
!
   capj(i, it)= sum(orglambda(i, :)*capg(:, it))
   deriv_capj_wrt_caph(i, it)= sum(qnty3*capg(:, it))+capj(i, it)**2
!
   deriv_capj_wrt_beta(it, pplus1)= sum(qnty3*xnodes*capg(:, it))+ &
   capj(i, it)*sum(orglambda(i, :)*xnodes*capg(:, it))
   deriv_capj_wrt_beta(it, 1:p)= z(i, :)*deriv_capj_wrt_caph(i, it)

  do l=1, pplus1
   e_yt_deriv_capj_wrt_beta(it, l)= e_yt_deriv_capj_wrt_beta(it, l)+ &
   capy(i, it)*deriv_capj_wrt_beta(it, l)
  end do
!
! For calculation of the first part of \Sigma_1
  do l1=1, pplus1
   do l2=1, pplus1
    cap_sigma_1_1st(l1, l2)= cap_sigma_1_1st(l1, l2)+(capy(i, it)* &
    deriv_capj_wrt_beta(it, l1)*dcapht(it))*zstar(i, l2)
   end do
  end do
!
! Estimation of cap_d_1_zstar
    tempo100=(sum((orgderiv_lambda(i, :)- &
    orglambda(i, :)**2)*capg( :, it))+capj(i, it)**2)*capy(i, it)
! For cap_d_1
    cap_d_1(it)=cap_d_1(it)+tempo100
! For cap_d_1_zstar
```

```
    do l=1, pplus1
     cap_d_1_zstar(it, l)=cap_d_1_zstar(it, l)+tempo100*zstar(i, l)
    end do
! Calculation of cap_d_2
    cap_d_2(it, :)=cap_d_2(it, :)+sum((orglambda(i, :)- &
    capj(i, it))*pr_yt_x_z(i,  :)*capg(:, it))*swtheta(i, :)
    do l=1, ntheta
     cap_d_2(it, l)=cap_d_2(it, l)- sum((orglambda(i, :)- &
     capj(i, it))*pr_yt_x_z(i,  :)*store_sxtheta(i, :, l)*capg( :, it))
    end do
! Calculation of cap_d_2_star
    do l1=1, pplus1
     do l2=1, ntheta
      tempo_mat(l1, l2)=swtheta(i, l2)*zstar(i, l1)
     end do
    end do
    cap_d_2_zstar(it, :, :)=cap_d_2_zstar(it, :, :)+sum((orglambda(i,  :)- &
    capj(i, it))*pr_yt_x_z(i,  :)*capg( :, it))*tempo_mat
    do l=1, ntheta
     cap_d_2_zstar(it,  :, l)=cap_d_2_zstar(it,  :, l)- &
      sum((orglambda(i, :)- capj(i, it))*pr_yt_x_z(i, :)* &
     store_sxtheta(i, :,  l)*capg(:, it))*zstar(i, :)
    end do
!  The following part is for the calculation of cap_q_1 and cap_q_zstar
    do j=1, n
!
    call densityofxvecgivenz(nnodes, den, ntheta, p, theta, (wnodes-v(i)),&
     z(j, :))
!
      new_tempo_eta=exp(eta(j)+beta2*(wnodes-v(i))+capht(it))
      if(r.eq.0.d0) then
       new_tempo_lambda=new_tempo_eta
       new_tempo_caplambda=new_tempo_eta
      else
       new_tempo_lambda=1/(r+(1/new_tempo_eta))
       new_tempo_caplambda=(1/r)*log(1+ r*new_tempo_eta)
      endif
!
      tempo400=weight*(pr_yt_w_z(j, :)*(new_tempo_lambda- &
```

```
      new_capj(j, :))*exp(-new_tempo_caplambda)*den/denominator(j, :)&
      )/sum(weight*den)
!
      sum_tempo400=sum(tempo400)
!
     cap_q_1(i, it)=cap_q_1(i, it)+ sum_tempo400
     cap_q_zstar(i, it, 1:p)=cap_q_zstar(i, it, 1:p)+zstar(j, 1:p)*sum_tempo400
     cap_q_zstar(i, it, pplus1)=cap_q_zstar(i, it, pplus1)+sum(tempo400*wnodes)
!
!
     tempo500=(qnty200(j, :)*den*weight)!/sum(den*weight)
     mean_tempo500=sum(tempo500)
!
     cap_q_1(i, it)=cap_q_1(i, it)-mean_tempo500
     cap_q_zstar(i, it, 1:p)=cap_q_zstar(i, it, 1:p)-z(j, 1:p)*mean_tempo500
     cap_q_zstar(i, it, pplus1)=cap_q_zstar(i, it, pplus1)-sum(tempo500*wnodes)
    end do  ! for the loop of j
!
! print*, 'i= ', i
  end do
 end do  ! end for the loop of it

 do it=1, nofail
  do l=1, pplus1
    tempo900(it, l)=sum(capy(:, it)*zstar(:, l)*capj(:, it))/float(n)
  end do
 end do
!  print*, ' \Sigma_1 =' , cap_sigma_1,  ' cap_phi(1, :, :)= ',  cap_phi(1, :, :)
! This is after the end of the loop for i
 e_yt_deriv_capj_wrt_beta=e_yt_deriv_capj_wrt_beta/float(n)
! Estimation of $C_D(u)$ and $C_N(u)$
 do k=1, nofail
  cd(k)=sum(capj(:, k)*capy(:, k))/float(n)
  cn(k)=sum(deriv_capj_wrt_caph(:, k)*capy(:, k))/float(n)
 end do
 ratio_cn_2_cd=cn/cd
! Estimation of $\lambda^{*}\{H_0(t)\}$
 do k=1, nofail
  lambda_star_caph(k)= exp(sum(ratio_cn_2_cd(1:k)*dcapht(1:k)) )
```

```
 end do
! Estimation of \gamma_1(t) and gamma_2(t)
 do it=1, nofail
  do l=1, pplus1
   gamma_1(it, l)=-(1.d0/lambda_star_caph(it))* sum (lambda_star_caph(1:it)*&
   e_yt_deriv_capj_wrt_beta(1:it, l)*dcapht(1:it)/cd(1:it) )
!
   gamma_2(it, l)=-  (e_yt_deriv_capj_wrt_beta(it, l)+cn(it)*gamma_1(it, l))*&
   dcapht(it)/cd(it)
  end do
 end do
!
! The first part of \Sigma_1  is calculated within the loop for i
 ! The second part of \Sigma_1
  do l1=1, pplus1
   do l2=1, pplus1
    do i=1, n
     cap_sigma_1_2nd(l1, l2)=cap_sigma_1_2nd(l1, l2)+ sum(capy(i, :)*  &
     deriv_capj_wrt_caph(i, :)*gamma_1(:, l1)*dcapht)*zstar(i, l2)
    end do
   end do
  end do
! The third part of \Sigma_1
   do l1=1, pplus1
    do l2=1, pplus1
     do i=1, n
      cap_sigma_1_3rd(l1, l2)=cap_sigma_1_3rd(l1, l2)+(sum(capy(i, :)*&
      capj(i, :)*gamma_2(:, l1))*zstar(i, l2))
     end do
    end do
   end do
   cap_sigma_1_1st=cap_sigma_1_1st/float(n)
   cap_sigma_1_2nd=cap_sigma_1_2nd/float(n)
   cap_sigma_1_3rd=cap_sigma_1_3rd/float(n)
! The final form of estimated cap_sigma_1 or \Sigma_1
  cap_sigma_1=cap_sigma_1_1st+cap_sigma_1_2nd+cap_sigma_1_3rd
!
! Estimation of $\Phi_i(u)$
   do it=1, nofail
```

```
    qnty7=0.d0
    do j=1, n
     qnty7=qnty7+sum(capy(j, it:nofail)*deriv_capj_wrt_caph(j,it:nofail)*&
     dcapht(it:nofail)/lambda_star_caph(it:nofail))*zstar(j, :)
     qnty9(j)=sum(capy(j, it:nofail)*capj(j, it:nofail)*dcapht(it:nofail)*&
     cn(it:nofail)/(lambda_star_caph(it:nofail)*cd(it:nofail)))
    end do
    qnty7=(lambda_star_caph(it)/cd(it))*qnty7/float(n)
    do l=1, pplus1
     qnty8(l)=sum(capy(:, it)*capj(:, it)*zstar(:, l))
    end do
    qnty8=qnty8/(float(n)*cd(it))
    do l=1, pplus1
     qnty10(l)=sum(qnty9*zstar(:, l))
    end do
    qnty10=(lambda_star_caph(it)/cd(it))*qnty10/float(n)
    qnty11=-qnty7-qnty8+qnty10
    do i=1, n
     do l1=1, pplus1
      do l2=1, pplus1
       sigma_star_1st(l1, l2)=sigma_star_1st(l1, l2)+ (zstar(i, l1)+&
       qnty11(l1))*(zstar(i, l2)+qnty11(l2))*capy(i, it)*capj(i, it)*dcapht(it)
      end do
!     print*, '66666666666666666666= ',  cap_phi(i, it, :)
     end do
    end do
   end do
    sigma_star_1st=sigma_star_1st/float(n)
!
! Estimation of \Upsilon
! Estimation of cap_d_1, note that the main calculation
! of cap_d_1 was done within the loop of i
  cap_d_1=cap_d_1/float(n)
! Estimation of cap_d_1_zstar, note that the main
! calculation of cap_d_1_zstar was done within the loop of i
  cap_d_1_zstar=cap_d_1_zstar/float(n)
!
! Estimation of cap_d_2
  final_cap_d_2=matmul(cap_d_2, inv_a_wnz)/float(n)
```

```
  cap_d_2= final_cap_d_2
!
!  Estimation of cap_d_2_zstar
    do it=1, nofail
     final_cap_d_2_zstar(it,:, :) = matmul(cap_d_2_zstar(it, :, :), inv_a_wnz)
    end do
    cap_d_2_zstar=final_cap_d_2_zstar/float(n)
!
  do i=1, n
   do it=1, nofail
    cap_q_1(i, it)=cap_q_1(i, it)/float(n)
    cap_q_1(i, it)=cap_q_1(i, it)+(1.d0/n)*dot_product(cap_d_2(it, :), &
    (swtheta(i, :)-intsw(i,:)) )
    cap_q_zstar(i, it, :)=cap_q_zstar(i, it, :)/float(n)
    cap_q_zstar(i, it, :)=cap_q_zstar(i, it, :)+&
    (1.d0/n)*matmul(cap_d_2_zstar(it, :, :), (swtheta(i, :)-intsw(i,:)) )
   end do
  end do
!
! Calculation of Upsilon
! Calculation of G(x|t, Z_i, W_i)
! Calculation of \Upsilon_i(u)
  do it=1, nofail
   tempo600(it)=sum(cap_d_1(1:it)*dcapht(1:it)/cd(1:it))
  end do
 tempo1000=0.d0
! print*, 'tempo1000= ', tempo1000(1:2, 1:2)
 do it=1, nofail
  do i=1, n
   tempo700=sum(exp(tempo600(1:it))*cap_q_1(i, 1:it)*dcapht(1:it)/cd(1:it))
   do l=1, pplus1
    tempo800= -cap_q_zstar(i, it, l)+cap_d_1_zstar(it, l)*exp(-tempo600(it))*&
    tempo700+ tempo900(it, l)*(cap_q_1(i, it)/cd(it)- (cap_d_1(it)/cd(it))*&
    tempo700*exp(-tempo600(it)) )
    tempo1000(i, l)=tempo1000(i, l)+dcapht(it)*tempo800
!     print*, 'tempo800= ',  tempo800
   end do
  end do
 end do
```

```
!
   do l1=1, pplus1
    do l2=1, pplus1
       sigma_star_2nd(l1, l2)=sigma_star_2nd(l1, l2)+sum(tempo1000(:, l1)*&
       tempo1000(:, l2))
    end do
   end do
   sigma_star_2nd= sigma_star_2nd/float(n)
   sigma_star=sigma_star_1st+ sigma_star_2nd
!    print*, 'sigma_star= ', sigma_star
 return
 end subroutine
!*****************


    subroutine intswgivenz(h, intsw, n, nnodes, ntheta, p, theta, v, weight,&
    xnodes, z)
    implicit none
    integer:: n, nnodes,  ntheta, p
    real*8 :: h, intsw(n,ntheta), theta(ntheta), v(n), weight(nnodes),&
    xnodes(nnodes), z(n, p)
! local
    integer:: i, j, k, l
    real*8 :: den(nnodes), denx(nnodes), swthetanew(n, nnodes, ntheta), &
    sxtheta(nnodes,  ntheta), wnodes(nnodes)
    wnodes=xnodes

    do j=1, n
      call sxgivenz( nnodes, ntheta, p, sxtheta, theta, xnodes, z(j, :))

     do k=1, nnodes
      call densityofxvecgivenwnz(den, h,  n, nnodes, ntheta, p, theta, v, &
      wnodes(k), weight, xnodes, z(j, :))
      do l=1, ntheta
       swthetanew(j, k, l)=sum(sxtheta(:, l)*den)
      end do
     end do
    end do
    do i=1, n
```

```
     intsw(i, :)=0.d0
     do j=1, n
      call densityofxvecgivenz(nnodes, denx, ntheta, p, theta, &
      (xnodes-v(i)), z(j, :))
      do l=1, ntheta
       intsw(i, l)=intsw(i, l)+ sum(swthetanew(j, :, l)*denx*weight)/&
       sum(denx*weight)
      end do
     end do
    end do
    intsw=intsw/float(n)
    return
    end subroutine
!
!
! This subroutine calculates the score function for theta based on the latent x
    subroutine sxgivenz( nx, ntheta, p, sxtheta, theta, xvec, z0)
    implicit none
    integer::  nx, ntheta, p
    real*8 :: sxtheta(nx,ntheta), theta(ntheta), xvec(nx),  z0(p)
! local
    integer:: i, l, pplus1
    real*8:: tempo(nx)
    pplus1=p+1

    tempo=(xvec-theta(1)-dot_product(theta(2:pplus1), z0))
     sxtheta(:,  1)=tempo/theta(ntheta)
     do l=2, pplus1
      sxtheta(:, l)=tempo*z0(l-1)/theta(ntheta)
     end do
     sxtheta(:,  ntheta)=-0.5d0/theta(ntheta)+0.5d0*tempo*tempo/&
     (theta(ntheta)*theta(ntheta))

    return
    end subroutine
!
!
!
!
```

```
! This subroutine calculates the score function for theta based on the observed W
    subroutine swgivenz(h, n, neww, nnodes, ntheta, p, swtheta, theta, v, &
    weight, xnodes, z)
    implicit none
    integer:: n, nnodes, ntheta, p
    real*8 :: h, neww(n), swtheta(n, ntheta), theta(ntheta), v(n),&
    weight(nnodes), xnodes(nnodes), z(n, p)
! local
    integer:: i, l
    real*8 ::  den(nnodes), sxtheta(nnodes,  ntheta)

    do i=1, n
     call densityofxvecgivenwnz(den, h,  n, nnodes, ntheta, p, theta, v, &
      neww(i), weight, xnodes, z(i, :))

     call sxgivenz( nnodes, ntheta, p, sxtheta, theta, xnodes, z(i, :))

     do l=1, ntheta
      swtheta(i, l)=sum(sxtheta(:, l)*den)
     end do
    end do
    return
    end subroutine
!
!*****************
!!  The following function evaluates the density of U at x0, where W=X+U.
   real*8 function densityofu(h, nv, v, x0)
   implicit none
   integer:: nv
   real*8 :: h, v(nv), x0
!  local
   real*8 :: pi
   pi=22.d0/7.d0
    densityofu=sum(exp(-0.5d0*(x0-v)**2/h**2))/(nv*h*sqrt(2*pi))
   return
   end
!
!
!  The following function evaluates the density of X at X=x0
```

```
!  given Z=z0 i.e., f_{X|Z, \theta}(x0|z0)
   real*8 function densityofxgivenz(ntheta, p, theta, x0, z0)
   implicit none
   integer ::  ntheta, p
   real*8 ::  theta(ntheta), x0,  z0(p)
! local variables
   real*8 ::  pi
    pi=22.d0/7.d0
    densityofxgivenz=exp(-0.5*(x0-theta(1)-dot_product(z0, &
    theta(2:(p+1))))**2/theta(ntheta))/sqrt(2*pi*theta(ntheta))
   return
   end function
!
!
!  The following function evaluates the density of X at
!  X=xnodes given Z=z0 i.e., f_{X|Z, \theta}(xnodes|z0)
   subroutine densityofxvecgivenz(capl, den, ntheta, p, theta, xvec, z0)
   implicit none
   integer ::  capl, ntheta, p
   real*8 ::  den(capl), theta(ntheta), xvec(capl),  z0(p)
! local variables
   real*8 ::  pi
    pi=22.d0/7.d0
    den=exp(-0.5*(xvec-theta(1)-dot_product(z0, theta(2:(p+1))) )**2/&
    theta(ntheta))/sqrt(2*pi*theta(ntheta))
   return
   end subroutine
!

!   Evaluates the density of W at W0 given Z=z0
    real*8 function densityofwgivenz(h, n, nnodes, ntheta, p,  theta, v, &
    w0, weight, xnodes, z0)
    implicit none
    integer :: n, nnodes, ntheta, p
    real*8 :: h, theta(ntheta), v(n), w0, weight(nnodes), xnodes(nnodes), z0(p)
!   local
    integer:: k
    real*8 :: densityofu, densityofxgivenz
     densityofwgivenz=0.d0
```

```
     do k=1, nnodes
      densityofwgivenz=densityofwgivenz+ densityofxgivenz(ntheta, p, theta,&
      xnodes(k), z0)*densityofu(h, n, v, (w0-xnodes(k)))*weight(k)
     end do
     return
     end function
!

!   Evaluates the density of W at number of points
!  (W01, W02, \cdots, W0L) given Z=z0
    subroutine densityofwvecgivenz(capl, den, h,  n, nnodes, ntheta, p,&
    theta, v, wvec, weight, xnodes, z0)
    implicit none
    integer :: capl, n, nnodes, ntheta, p
    real*8 :: den(capl), h, theta(ntheta), v(n), wvec(capl),&
    weight(nnodes), xnodes(nnodes), z0(p)
!   local
    integer:: i, k
    real*8 :: densityofu, densityofxgivenz, store_densityofxgivenz(nnodes),&
    temp(nnodes)
     den=0.d0
     do k=1, nnodes
      store_densityofxgivenz(k)= densityofxgivenz(ntheta, p, theta,  &
      xnodes(k), z0)
     end do
     do i=1, capl
      do k=1, nnodes
       temp(k)=densityofu(h, n, v, (wvec(i)-xnodes(k)))
       den(i)=den(i)+store_densityofxgivenz(k)*temp(k)  &
       *weight(k)
      end do
      den=den/sum(store_densityofxgivenz*temp*weight )
     end do
     return
     end subroutine
!!
!   Evaluates the density of X at  X=xnodes given W=W0
!   and Z=z0 (note that xnodes is a vector)
    subroutine densityofxvecgivenwnz(den, h, n, nnodes, ntheta, p, theta,&
```

```
    v, w0, weight, xnodes, z0)
    implicit none
    integer :: n, nnodes, ntheta, p
    real*8 :: den(nnodes), h, theta(ntheta), v(n), w0, weight(nnodes),&
    xnodes(nnodes), z0(p)
!   local
    integer:: k
    real*8 :: densityofu, densityofxgivenz, store_densityofu(nnodes),&
    store_densityofxgivenz(nnodes), tempo_sum
     den=0.d0
     do k=1, nnodes
      store_densityofxgivenz(k)= densityofxgivenz(ntheta, p, theta,  &
      xnodes(k), z0)
      store_densityofu(k)=densityofu(h, n, v, (w0-xnodes(k)))
     end do
     tempo_sum=sum(store_densityofxgivenz*store_densityofu*weight)
     den=store_densityofxgivenz*store_densityofu*weight/tempo_sum
     return
     end subroutine
!*!
   subroutine neglkfunc(capy, gamma, lglk, n, nnodes,  p, pplus2, storeden,&
   xnodes, z)
   implicit none
   integer:: n, nnodes,  p, pplus2
   real*8::  capy(n), gamma(pplus2), lglk, storeden(n, nnodes), &
   xnodes(nnodes), z(n, p)
! local
   integer:: i, pplus1
   real*8 :: tmp_eta(nnodes), tmp_prob(nnodes), tmp_prob_bar(nnodes),&
    prob, prob_bar
    pplus1=p+1
    lglk=0.d0
    do i=1, n
     tmp_eta=gamma(1)+dot_product(z(i, :), gamma(2:(pplus1)))+&
     xnodes*gamma(pplus2)
     tmp_prob=1.d0/(1.d0+exp(-tmp_eta))
     tmp_prob_bar=1-tmp_prob
     prob=sum(tmp_prob*storeden(i, :) )
     prob_bar=sum(tmp_prob_bar*storeden(i, :))
```

```
     if(prob.eq.1.d0) prob=0.999999999d0
     if(prob_bar.eq.0.d0) prob_bar=1E-10
     lglk=lglk-capy(i)*log(prob)+(1.d0-capy(i))*log(prob_bar)
    ! if(lglk.ne.lglk) print*, prob, prob_bar
    end do

   return
   end subroutine
! ------------------------------------------------------------------
       SUBROUTINE gaussj(a,n,np)!
       implicit none
       INTEGER*4 m,mp,n,np,NMAX
       DOUBLE PRECISION a(np,np)!
       PARAMETER (NMAX=50)
!      Linear equation solution by Gauss-Jordan elimination,
!      equation (2.1.1) above.
!      a(1:n,1:n) is an input matrix stored in an array of
!      physical dimensions np by np.
!      b(1:n,1:m) is an input matrix containing the
!      m right-hand side vectors,
!      stored in an array of physical dimensions np by mp.
!      On output, a(1:n,1:n) is
!      replaced by its matrix inverse, and b(1:n,1:m) is
!      replaced by the corresponding
!      set of solution vectors. Parameter: NMAX is the largest
!      anticipated value of n.
        INTEGER*4 i,icol,irow,j,k,l,ll,indxc(NMAX),indxr(NMAX), &
       ipiv(NMAX)   ! The integer arrays ipiv, indxr, and indxc are used
         double precision big,dum,pivinv    !for bookkeeping on the pivoting.
        do 11 j=1,n
          ipiv(j)=0
 11     continue
         do 22 i=1,n !This is the main loop over the columns to be reduced.
          big=0.0d0
          do 13 j=1,n! This is the outer loop of the search for aivot element.
            if(ipiv(j).ne.1) then
              do 12 k=1,n
               if (ipiv(k).eq.0) then
                 if (abs(a(j,k)).ge.big)then
```

```
                 big=abs(a(j,k))
                 irow=j
                 icol=k
               endif
             endif
 12         continue
           endif
 13      continue
         ipiv(icol)=ipiv(icol)+1
!       We now have the pivot element, so we interchange rows,
!       if needed, to put the pivot element on the diagonal. The columns
!       are not physically interchanged, only relabeled:
!        indxc(i), the column of the ith pivot element, is the ith column that
!       is reduced, while indxr(i) is the row in which that pivot element
!       was originally located.
!      If indxr(i)  = indxc(i) there is an implied column
!       interchange. With this form of bookkeeping, the
!       solution b s will end up in the correct order, and
!       the inverse matrix will be scrambled by columns.
        if (irow.ne.icol) then
          do 14 l=1,n
              dum=a(irow,l)
              a(irow,l)=a(icol,l)
              a(icol,l)=dum
 14       continue
!          do 15 l=1,m
!              dum=b(irow,l)
!              b(irow,l)=b(icol,l)
!              b(icol,l)=dum
! 15       continue
        endif
        indxr(i)=irow   ! We are now ready to divide the pivot
        indxc(i)=icol   !row by the pivot element, located at irow and icol.
        if (a(icol,icol).eq.0.) return! pause  'singular matrix in gaussj'
        pivinv=1.0d0/a(icol,icol)
        a(icol,icol)=1.0d0
        do 16 l=1,n
            a(icol,l)=a(icol,l)*pivinv
 16     continue
```

```
!         do 17 l=1,m
!             b(icol,l)=b(icol,l)*pivinv
! 17      continue
         do 21 ll=1,n               ! Next, we reduce the rows...
              if(ll.ne.icol)then    !...except for the pivot one, of course.
                  dum=a(ll,icol)
                  a(ll,icol)=0.0d0
                  do 18 l=1,n
                      a(ll,l)=a(ll,l)-a(icol,l)*dum
 18               continue
!                  do 19 l=1,m
!                      b(ll,l)=b(ll,l)-b(icol,l)*dum
! 19               continue
              endif
 21       continue
 22     continue ! This is the end of the main
!                ! loop over columns of the reduction.
        do 24 l=n,1,-1        !It only remains to unscramble the
!                             !solution in view of the column interchanges.
           if(indxr(l).ne.indxc(l))then      !We do this by
!                                            !interchanging pairs of columns
!        in the reverse order that the permutation was built up.
               do 23 k=1,n
                  dum=a(k,indxr(l))
                  a(k,indxr(l))=a(k,indxc(l))
                  a(k,indxc(l))=dum
 23            continue
           endif
 24     continue
        return                      ! And we are done.
        END
! ------------
```