Biostatistics (2019), **0**, 0, *pp.* 1–5
doi:10.1093/biostatistics/supplementary

Supplementary Material for "Matched Case-Control Data with a Misclassified Exposure: What can be done with Instrumental Variables?"

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A.1. Identification of the parameters of the model $\text{pr}(W = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z})$

The identification comes from the assumed non-linear structure for $\text{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z})$. Had $\text{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z})$ been linear, the parameters would not be identifiable. In short we write $H(\gamma_0+\boldsymbol{\gamma}_1^T\boldsymbol{S}+\boldsymbol{\gamma}_2^T\boldsymbol{X}^*+\boldsymbol{\gamma}_3^T\boldsymbol{Z})$ as $H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})$. In our case $H(\cdot)$ is the logistic function, which is nonlinear.

To see the identifiability issue, we need to show that for every given parameter set $(\boldsymbol{\gamma}, \alpha_0, \alpha_1)$ if another parameter set $(\boldsymbol{\gamma}^*, \alpha_0^*, \alpha_1^*)$ satisfies $\text{pr}(W = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z}; \alpha_0, \alpha_1, \boldsymbol{\gamma}) = \text{pr}(W = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z}; \alpha_0^*, \alpha_1^*, \boldsymbol{\gamma}^*)$ for every choice of $\boldsymbol{S}$, $\boldsymbol{X}^*$ and $\boldsymbol{Z}$, then $(\boldsymbol{\gamma}^*, \alpha_0^*, \alpha_1^*) = (\boldsymbol{\gamma}, \alpha_0, \alpha_1)$. To see this, by Equation (3.3) we start with

$$\alpha_0 + (1 - \alpha_0 - \alpha_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) = \alpha_0^* + (1 - \alpha_0^* - \alpha_1^*)H(\boldsymbol{\gamma}^*, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) \tag{A.1}$$

for every choice of $(\boldsymbol{S}^T, \boldsymbol{X}^{*,T}, \boldsymbol{Z}^{*,T})^T$. Let $\boldsymbol{\gamma}^* = -\boldsymbol{\gamma}$, $\alpha_0^* = 1 - \alpha_1$ and $\alpha_1^* = 1 - \alpha_0$. Then

$H(\boldsymbol{\gamma}^*, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) = H(-\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) = 1 - H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})$ and

$$\begin{aligned}
\alpha_0^* + (1 - \alpha_0^* - \alpha_1^*)H(\boldsymbol{\gamma}^*, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) &= (1 - \alpha_1) + (1 - 1 + \alpha_1 - 1 + \alpha_0)H(-\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) \\
&= (1 - \alpha_1) + (-1 + \alpha_0 + \alpha_1)\{1 - H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})\} \\
&= (1 - \alpha_1) + (-1 + \alpha_0 + \alpha_1) - (-1 + \alpha_0 + \alpha_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) \\
&= \alpha_0 + (1 - \alpha_0 - \alpha_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}).
\end{aligned}$$

On the other hand, under the monotonicity restriction $\alpha_0+\alpha_1 < 1$, if $\alpha_1^* = 1-\alpha_0$ and $\alpha_0^* = 1-\alpha_1$, then $\alpha_0^* + \alpha_1^* = (1 - \alpha_1 + 1 - \alpha_0) = 1 + (1 - \alpha_0 - \alpha_1) > 1$. Hence, this particular choice of α_0^*, α_1^* does not satisfy the restriction, and is not a cause of concern anymore.

Finally, we need to check if there is any other choice of $(\alpha_0^*, \alpha_1^*, \boldsymbol{\gamma}^*)$ that satisfies (A.1). Suppose that there exists $(\alpha_0^*, \alpha_1^*, \boldsymbol{\gamma}^*)$ that satisfies (A.1) for every choice of $\boldsymbol{S}$, $\boldsymbol{X}^*$ and $\boldsymbol{Z}$. This implies that for every $(\boldsymbol{S}_k, \boldsymbol{X}_k^*, \boldsymbol{Z}_k)$, $k = 1, 2, \ldots,$

$$\alpha_0^* + (1 - \alpha_0^* - \alpha_1^*)H(\boldsymbol{\gamma}^*, \boldsymbol{S}_k, \boldsymbol{X}_k^*, \boldsymbol{Z}_k) = \alpha_0 + (1 - \alpha_0 - \alpha_1)H(\boldsymbol{\gamma}, \boldsymbol{S}_k, \boldsymbol{X}_k^*, \boldsymbol{Z}_k).$$

Since $1-\alpha_0^*-\alpha_1^* > 0$ and $1-\alpha_0-\alpha_1 > 0$, it is readily seen that each element of $(\boldsymbol{\gamma}_1^*, \boldsymbol{\gamma}_2^*)$ must have the same sign as the corresponding element of $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$. By letting $T = \gamma_0+\boldsymbol{\gamma}_1^T\boldsymbol{S}+\boldsymbol{\gamma}_2^T\boldsymbol{X}^*+\boldsymbol{\gamma}_3^T\boldsymbol{Z} \to -\infty$ (and then $T^* = \gamma_0^* + \boldsymbol{\gamma}_1^{*T}\boldsymbol{S} + \boldsymbol{\gamma}_2^{*T}\boldsymbol{X}^* + \boldsymbol{\gamma}_3^T\boldsymbol{Z} \to -\infty$ also), it is clear that $\alpha_0^* = \alpha_0$. Likewise, due to the nonlinearity of $H(\cdot)$, $\alpha_1^* = \alpha_1$. This leads to $T^* = T$ and thus $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$, showing the identifiability of these parameters.

A.2. Proof of Lemma 1

Because of the logistic model assumption and the assumption on W and $\boldsymbol{X}^*$ we can write

$$\begin{aligned}
1 - \mathrm{pr}(Y = 0|\boldsymbol{S}, W, X, \boldsymbol{X}^*, \boldsymbol{Z}) &= \mathrm{pr}(Y = 1|\boldsymbol{S}, W, X, \boldsymbol{X}^*, \boldsymbol{Z}) \\
&= \mathrm{pr}(Y = 1|\boldsymbol{S}, X, \boldsymbol{Z}) \\
&= \exp\{g_0(\boldsymbol{S}) + \beta_1 X + \boldsymbol{\beta}_2^T\boldsymbol{Z}\}\mathrm{pr}(Y = 0|\boldsymbol{S}, X, \boldsymbol{Z}),
\end{aligned}$$

where $g_0(\cdot)$ is given in Model (2.1). We now consider

$$\begin{aligned}
&\text{pr}(Y=1|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\\
&=\sum_{x=0,1}\text{pr}(Y=1|\boldsymbol{S},W,X=x,\boldsymbol{X}^*,\boldsymbol{Z})\text{pr}(X=x|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\\
&=\sum_{x=0,1}\text{pr}(Y=1|\boldsymbol{S},X=x,\boldsymbol{Z})\text{pr}(X=x|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\\
&=\sum_{x=0,1}\exp\{g_0(\boldsymbol{S}_i)+\beta_1 x+\boldsymbol{\beta}_2^T\boldsymbol{Z}\}\text{pr}(Y=0|\boldsymbol{S},X=x,\boldsymbol{Z})\text{pr}(X=x|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\\
&=\sum_{x=0,1}\exp\{g_0(\boldsymbol{S}_i)+\beta_1 x+\boldsymbol{\beta}_2^T\boldsymbol{Z}\}\text{pr}(X=x|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\text{pr}(Y=0|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\\
&=\text{pr}(Y=0|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\sum_{x=0,1}\exp\{g_0(\boldsymbol{S})+\beta_1 x+\boldsymbol{\beta}_2^T\boldsymbol{Z}\}\text{pr}(X=x|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\\
&=\text{pr}(Y=0|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\exp\{g_0(\boldsymbol{S})+\boldsymbol{\beta}_2^T\boldsymbol{Z}\}\{\exp(\beta_1)\text{pr}(X=1|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\\
&\quad+\text{pr}(X=0|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\}\\
&\equiv\text{pr}(Y=0|\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z})\exp\{g_0(\boldsymbol{S})+\boldsymbol{\beta}_2^T\boldsymbol{Z}+g_1(\beta_1,\boldsymbol{S}_i,W,\boldsymbol{X}^*,\boldsymbol{Z},\boldsymbol{\gamma},\boldsymbol{\eta})\},
\end{aligned}$$

where the expression of $g_1(\beta_1,\boldsymbol{S},W,\boldsymbol{X}^*,\boldsymbol{Z},\boldsymbol{\gamma},\boldsymbol{\eta})$ is obtained after plugging the expression for $\text{pr}(X=1|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})$ and $\text{pr}(X=0|\boldsymbol{S},W,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})$ from Equations (3.4) and (3.5). In particular,

$$\begin{aligned}
&\exp\{g_1(\beta_1,\boldsymbol{S},W=1,\boldsymbol{X}^*,\boldsymbol{Z},\boldsymbol{\gamma},\boldsymbol{\eta})\}\\
&=\exp(\beta_1)\text{pr}(X=1|\boldsymbol{S},W=1,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})+\text{pr}(X=0|\boldsymbol{S},W=1,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\\
&=\exp(\beta_1)\frac{(1-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}{\alpha_0+(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}+1-\frac{(1-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}{\alpha_0+(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}\\
&=\frac{\exp(\beta_1)(1-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})+\alpha_0\{1-H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\}}{\alpha_0+(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})},
\end{aligned}\tag{A.2}$$

$$\begin{aligned}
&\exp\{g_1(\beta_1,\boldsymbol{S},W=0,\boldsymbol{X}^*,\boldsymbol{\gamma},\boldsymbol{\eta})\}\\
&=\exp(\beta_1)\text{pr}(X=1|\boldsymbol{S},W=0,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})+\text{pr}(X=0|\boldsymbol{S},W=0,\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\\
&=\exp(\beta_1)\frac{\alpha_1 H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}{1-\alpha_0-(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}+1-\frac{\alpha_1 H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}{1-\alpha_0-(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}\\
&=\frac{\exp(\beta_1)\alpha_1 H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})+(1-\alpha_0)\{1-H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\}}{1-\alpha_0-(1-\alpha_0-\alpha_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})}.
\end{aligned}\tag{A.3}$$

A.3. Proof of Theorem 1

Collecting $\boldsymbol{S}_\gamma(\boldsymbol{\gamma},\boldsymbol{\eta}), \boldsymbol{S}_\eta(\boldsymbol{\gamma},\boldsymbol{\eta}), S_{\beta_1}(\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta}), \boldsymbol{S}_{\boldsymbol{\beta}_2}(\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\eta})$ together and letting $\boldsymbol{\theta} = (\boldsymbol{\gamma}^T, \boldsymbol{\eta}^T, \beta_1, \boldsymbol{\beta}_2^T)^T$ and $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^T, \widehat{\boldsymbol{\eta}}^T, \widehat{\beta}_1, \widehat{\boldsymbol{\beta}}_2^T)^T$, we can write

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = A^{-1}\sum_{i=1}^{n} \boldsymbol{U}_i + o_p(1),$$

where $\boldsymbol{U}_i's$ are iid and mean zero and finite variance random vectors. $A = -E(\partial \boldsymbol{U}_i/\partial\boldsymbol{\theta})$. By the Central Limit Theorem we obtain the asymptotic normality of $\widehat{\boldsymbol{\theta}}$, and the asymptotic variance of $\sqrt{n}\widehat{\boldsymbol{\theta}}$ is $A^{-1}\mathrm{var}(\boldsymbol{U}_1)A^{-T}$. This asymptotic variance can be consistently estimated by $\widehat{A}^{-1}(\sum_{i=1}^n \widehat{\boldsymbol{U}}_i\widehat{\boldsymbol{U}}_i^T/n)\widehat{A}^{-T}$ with $\widehat{A} = -(1/n)\sum_{i=1}^n \partial\widehat{\boldsymbol{U}}_i/\partial\boldsymbol{\theta}$ and $\widehat{\boldsymbol{U}}_i$ being $\boldsymbol{U}_i$ with $\boldsymbol{\theta}$ replaced by $\widehat{\boldsymbol{\theta}}$.

A.4. Proof of Lemma 2

Part i) of Lemma 2

$$\begin{aligned}
\mathrm{pr}(Y=1|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) &= \sum_x \mathrm{pr}(Y=1|\boldsymbol{S},X=x,\boldsymbol{X}^*,\boldsymbol{Z})\mathrm{pr}(X=x|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) \\
&= \sum_x \exp\{g_0(\boldsymbol{S}) + \beta_1 x + \boldsymbol{\beta}_2^T\boldsymbol{Z}\}\mathrm{pr}(Y=0|\boldsymbol{S},X=x,\boldsymbol{X}^*,\boldsymbol{Z})\mathrm{pr}(X=x|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) \\
&= \sum_x \exp\{g_0(\boldsymbol{S}) + \beta_1 x + \boldsymbol{\beta}_2^T\boldsymbol{Z}\}\mathrm{pr}(X=x|\boldsymbol{S},\boldsymbol{X}^*,Y=0,\boldsymbol{Z})\mathrm{pr}(Y=0|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) \\
&= \mathrm{pr}(Y=0|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})[\exp\{g_0(\boldsymbol{S}) + \boldsymbol{\beta}_2^T\boldsymbol{Z}\}\{1 - H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\} \\
&\quad + \exp\{g_0(\boldsymbol{S}) + \beta_1 + \boldsymbol{\beta}_2^T\boldsymbol{Z}\}H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})] \\
&= \mathrm{pr}(Y=0|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\exp\{g_0(\boldsymbol{S}) + \boldsymbol{\beta}_2^T\boldsymbol{Z}\} \\
&\quad \times\{1 - H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) + \exp(\beta_1)H(\boldsymbol{\gamma},\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\}.
\end{aligned}$$

This implies

$$\mathrm{pr}(Y=1|\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z}) = H\{g_0(\boldsymbol{S}) + \boldsymbol{\beta}_2^T\boldsymbol{Z} + g_2(\boldsymbol{\gamma},\beta_1,\boldsymbol{S},\boldsymbol{X}^*,\boldsymbol{Z})\},$$

where

$$g_2(\boldsymbol{\gamma}, \beta_1, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) = \log\{1 - H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) + \exp(\beta_1) H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})\}.$$

Part ii) of Lemma 2

$$\begin{aligned}
\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}, Y = 1) &= \frac{\mathrm{pr}(Y = 1|\boldsymbol{S}, X = 1, \boldsymbol{X}^*, \boldsymbol{Z})\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{\mathrm{pr}(Y = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})} \\
&= \frac{\exp\{g_0(\boldsymbol{S}) + \beta_1 + \boldsymbol{\beta}_2^T \boldsymbol{Z}\}\mathrm{pr}(Y = 0|\boldsymbol{S}, X = 1, \boldsymbol{X}^*, \boldsymbol{Z})\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{\mathrm{pr}(Y = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})} \\
&= \frac{\exp\{g_0(\boldsymbol{S}) + \beta_1 + \boldsymbol{\beta}_2^T \boldsymbol{Z}\}\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 0, \boldsymbol{Z})\mathrm{pr}(Y = 0|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{\mathrm{pr}(Y = 1|\boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})} \\
&= \frac{\exp\{g_0(\boldsymbol{S}) + \beta_1 + \boldsymbol{\beta}_2^T \boldsymbol{Z}\}H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{\exp\{g_0(\boldsymbol{S}) + \boldsymbol{\beta}_2^T \boldsymbol{Z} + g_2(\boldsymbol{\gamma}, \beta_1, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})\}} \\
&= \frac{\exp(\beta_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{\exp\{g_2(\boldsymbol{\gamma}, \beta_1, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})\}} \\
&= \frac{\exp(\beta_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})}{1 - H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z}) + \exp(\beta_1)H(\boldsymbol{\gamma}, \boldsymbol{S}, \boldsymbol{X}^*, \boldsymbol{Z})} \\
&= \frac{\exp(\gamma_0 + \beta_1 + \boldsymbol{\gamma}_1^T \boldsymbol{S} + \boldsymbol{\gamma}_2^T \boldsymbol{X}^* + \boldsymbol{\gamma}_3^T \boldsymbol{Z})}{1 + \exp(\gamma_0 + \beta_1 + \boldsymbol{\gamma}_1^T \boldsymbol{S} + \boldsymbol{\gamma}_2^T \boldsymbol{X}^* + \boldsymbol{\gamma}_3^T \boldsymbol{Z})} \\
&= H(\gamma_0 + \beta_1 + \boldsymbol{\gamma}_1^T \boldsymbol{S} + \boldsymbol{\gamma}_2^T \boldsymbol{X}^* + \boldsymbol{\gamma}_3^T \boldsymbol{Z}).
\end{aligned}$$

Part iii) of Lemma 2

$$\begin{aligned}
&\mathrm{pr}(W = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
=&\mathrm{pr}(W = 1|\boldsymbol{S}, X = 0, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z})\mathrm{pr}(X = 0|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
&+\mathrm{pr}(W = 1|\boldsymbol{S}, X = 1, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z})\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
=&\mathrm{pr}(W = 1|X = 0)\mathrm{pr}(X = 0|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
&+\mathrm{pr}(W = 1|X = 1)\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
=&\alpha_0\{1 - \mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z})\} + (1 - \alpha_1)\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
=&\alpha_0 + (1 - \alpha_0 - \alpha_1)\mathrm{pr}(X = 1|\boldsymbol{S}, \boldsymbol{X}^*, Y = 1, \boldsymbol{Z}) \\
=&\alpha_0 + (1 - \alpha_0 - \alpha_1)H(\gamma_0 + \beta_1 + \boldsymbol{\gamma}_1^T \boldsymbol{S} + \boldsymbol{\gamma}_2^T \boldsymbol{X}^* + \boldsymbol{\gamma}_3^T \boldsymbol{Z}).
\end{aligned}$$