Web-based Supplementary Materials for: "Semiparametric approach for non-monotone missing covariates in a parametric regression model"

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These supplementary materials contain some results from the simulation study, some technical details for the identifiability issue, the regularity conditions, an explicit expression of the terms of the matrix D, and the proof of Theorem 1.

W-A1 Simulation study with non-ignorable missing data

Here the simulation design is the same as that in scenario 1 described in the main text with two partially missing variables X_1 and X_2. Missing data were created by following the two non-ignorable mechanisms, 1) $\text{logit}\{\text{pr}(R_k = 1|X_1, X_2, Y, Z)\} = 0.25Y + 0.25Z + X_2 + X_1$ and 2) $\text{logit}\{\text{pr}(R_k = 1|X_1, X_2, Y, Z)\} = 0.75 + Y + 0.25Z - X_1 + X_2$, for $k = 1, 2$. Although in both mechanisms R_k strongly depends on both X_1 and X_2, dependence on Y is weak and strong for mechanisms 1 and 2, respectively. Also, both mechanisms resulted in approximately 25% missing data for X_1 and for X_2. The results in Table W-1 show that the complete case method has significant bias in the parameter estimates. As expected, compared to the mean-score approach, the SP method shows much less bias in the estimates. The reason is that the SP method assumes NI- mechanism which allows R_1 to depend on X_2 along with Y and Z, and R_2 to depend on X_1 along with Y and Z – a relatively close model to the true missing mechanism than the MAR mechanism where R_k is assume not to depend on X_1 or X_2, for $k = 1, 2$. Note that for all methods, the bias also depends on how strongly the missingness mechanism depends on the response Y.

W-A2 Identifiability

Let $n_{x_1,x_2,y}$ be the number of observations with $X_1 = x_1$, $X_2 = x_2$, and $Y = y$ and $R_1 = R_2 = 1$, $m_{x_1,-,y}$ be the number observations with $X_1 = x_1$, missing X_2 and $Y = y$ (i.e., where $R_1 = 1$ and $R_2 = 0$), $m_{-,x_2,y}$ be the number observations with missing X_1, $X_2 = x_2$ and $Y = y$ (i.e., where $R_1 = 0$ and $R_2 = 1$), and $m_{-,-,y}$ be the number observations with missing X_1 and X_2 and $Y = y$ (i.e., where $R_1 = 0$ and $R_2 = 0$). Thus, $n_{0,0,0} + n_{0,0,1} + n_{0,1,0} + n_{0,1,1} + n_{1,0,0} + n_{1,0,1} + n_{1,1,0} + n_{1,1,1} + m_{-,0,0} + m_{-,0,1} + m_{-,1,0} + m_{-,1,1} + m_{0,-,0} + m_{0,-,1} + m_{1,-,0} + m_{1,-,1} + m_{-,-,0} + m_{-,-,1} =$

n. Let $u_{x_1,x_2} = \mathrm{pr}(X_1 = 1, X_2 = x_2)$, $v_{x_1,x_2} = \mathrm{pr}(Y = 1|X_1 = x_2, X_2 = x_2)$, and under the NI- mechanism, $\pi^{(1)}_{x_2,y} = \mathrm{pr}(R_1 = 1|X_2 = x_2, Y = y)$ and $\pi^{(2)}_{x_1,y} = \mathrm{pr}(R_2 = 1|X_1 = x_1, Y = y)$. Define $\theta = (u_{00}, u_{01}, u_{10}, v_{00}, v_{01}, v_{10}, v_{11}, \pi^{(1)}_{00}, \pi^{(1)}_{01}, \pi^{(1)}_{10}, \pi^{(1)}_{11}, \pi^{(2)}_{00}, \pi^{(2)}_{01}, \pi^{(2)}_{10}, \pi^{(2)}_{11})^T$. Observe that $E[\{\partial\log(L)/\partial\theta\}\{\partial\log(L)/\partial\theta\}^T]$ can be written as $A\mathrm{cov}(\tilde{n})A^T$ for some matrix A whose elements will be described below, and $\tilde{n} = (n_{0,0,0}, n_{0,0,1}, n_{0,1,0}, n_{0,1,1}, n_{1,0,0}, n_{1,0,1}, n_{1,1,0}, n_{1,1,1}, m_{-,0,0}, m_{-,0,1}, m_{-,1,0}, m_{-,1,1}, m_{0,-,0}, m_{0,-,1}, m_{1,-,0}, m_{1,-,1}, m_{-,-,0}, m_{-,-,1})^T$, a vector of random cell frequencies for a multinomial distribution with the total frequency n and the success probabilities, $p_{0,0,0} = \pi^{(1)}_{00}\pi^{(2)}_{00}u_{00}(1 - v_{00})$, $p_{0,1,0} = \pi^{(1)}_{10}\pi^{(2)}_{00}u_{01}(1 - v_{01})$, $p_{1,0,0} = \pi^{(1)}_{00}\pi^{(2)}_{10}u_{10}(1 - v_{10})$, $p_{1,1,0} = \pi^{(1)}_{10}\pi^{(2)}_{10}u_{11}(1 - v_{11})$, $p_{0,0,1} = \pi^{(1)}_{01}\pi^{(2)}_{01}u_{00}v_{00}$, $p_{0,1,1} = \pi^{(1)}_{11}\pi^{(2)}_{01}u_{01}v_{01}$, $p_{1,0,1} = \pi^{(1)}_{01}\pi^{(2)}_{11}u_{10}v_{10}$, $p_{1,1,1} = \pi^{(1)}_{11}\pi^{(2)}_{11}u_{11}v_{11}$, $p_{0,1,1} = \pi^{(1)}_{11}\pi^{(2)}_{01}u_{01}v_{01}$, $p_{-,0,0} = (1 - \pi^{(1)}_{00})\{\pi^{(2)}_{00}u_{00}(1 - v_{00}) + \pi^{(2)}_{10}u_{10}(1 - v_{10})\}$, $p_{-,0,1} = (1 - \pi^{(1)}_{01})(\pi^{(2)}_{01}u_{00}v_{00} + \pi^{(2)}_{11}u_{10}v_{10})$, $p_{-,1,0} = (1 - \pi^{(1)}_{10})\{\pi^{(2)}_{00}u_{01}(1 - v_{01}) + \pi^{(2)}_{10}u_{11}(1 - v_{11})\}$, $p_{-,1,1} = (1 - \pi^{(1)}_{11})(\pi^{(2)}_{01}u_{01}v_{01} + \pi^{(2)}_{11}u_{11}v_{11})$, $p_{0,-,0} = (1 - \pi^{(2)}_{00})\{\pi^{(1)}_{00}u_{00}(1 - v_{00}) + \pi^{(1)}_{10}u_{01}(1 - v_{01})$, $p_{0,-,1} = (1 - \pi^{(2)}_{01})\{\pi^{(1)}_{01}u_{00}v_{00} + \pi^{(1)}_{11}u_{01}v_{01}$, $p_{1,-,0} = (1 - \pi^{(2)}_{10})\{\pi^{(1)}_{00}u_{10}(1-v_{10}) + \pi^{(1)}_{10}u_{11}(1-v_{11})\}$, $p_{1,-,1} = (1-\pi^{(2)}_{11})(\pi^{(1)}_{01}u_{10}v_{10} + \pi^{(1)}_{11}u_{11}v_{11})$, $p_{-,-,0} = (1-\pi^{(1)}_{00})(1-\pi^{(2)}_{00})u_{00}(1-v_{00}) + (1-\pi^{(1)}_{10})(1-\pi^{(2)}_{00})u_{01}(1-v_{00}) + (1-\pi^{(1)}_{00})(1-\pi^{(2)}_{10})u_{10}(1-v_{10}) + (1-\pi^{(1)}_{10})(1-\pi^{(2)}_{10})u_{11}(1-v_{11})$, and $p_{-,-,1} = (1-\pi^{(1)}_{00})(1-\pi^{(2)}_{00})u_{00}v_{00} + (1-\pi^{(1)}_{10})(1-\pi^{(2)}_{00})u_{01}v_{00} + (1-\pi^{(1)}_{00})(1-\pi^{(2)}_{10})u_{10}v_{10} + (1-\pi^{(1)}_{10})(1-\pi^{(2)}_{10})u_{11}v_{11}$. The covariance matrix $\mathrm{cov}(\tilde{n})$ is positive semidefinite and has a rank of 17. Therefore, to prove that $A\mathrm{cov}(\tilde{n})A^T$ is nonsingular, according to Lemma 1 stated below, we just need to show that matrix A has full row rank. Observe that A is a 15×18 matrix. Suppose that a_j^T represents the jth row of matrix A for $j = 1, \ldots, 15$ with

$$\begin{aligned}
a_1^T\tilde{n} &= \frac{n_{000}}{u_{00}} + 0(n_{010}) + 0(n_{100}) - \frac{n_{110}}{u_{11}} + \frac{n_{001}}{u_{00}} + 0(n_{011}) + 0(n_{101}) - \frac{n_{111}}{u_{11}} \\
&+ m_{-,0,0}\frac{\pi^{(2)}_{00}(1-v_{00})}{u_{00}\pi^{(2)}_{00}(1-v_{00}) + u_{10}\pi^{(2)}_{10}(1-v_{10})} + m_{-,0,1}\frac{\pi^{(2)}_{01}v_{00}}{u_{00}\pi^{(2)}_{01}v_{00} + u_{10}\pi^{(2)}_{11}v_{10}} \\
&- m_{-,1,0}\frac{\pi^{(2)}_{10}(1-v_{11})}{u_{01}\pi^{(2)}_{00}(1-v_{01}) + u_{11}\pi^{(2)}_{10}(1-v_{11})} - m_{-,1,1}\frac{\pi^{(2)}_{11}v_{11}}{u_{01}\pi^{(2)}_{01}v_{01} + u_{11}\pi^{(2)}_{11}v_{11}} \\
&+ m_{0,-,0}\frac{\pi^{(1)}_{00}(1-v_{00})}{u_{00}\pi^{(1)}_{00}(1-v_{00}) + u_{01}\pi^{(1)}_{10}(1-v_{01})} + m_{0,-,1}\frac{\pi^{(1)}_{01}v_{00}}{u_{00}\pi^{(1)}_{01}v_{00} + u_{01}\pi^{(1)}_{11}v_{01}} \\
&- m_{1,-,0}\frac{\pi^{(1)}_{10}(1-v_{11})}{u_{10}\pi^{(1)}_{00}(1-v_{10}) + u_{11}\pi^{(1)}_{10}(1-v_{11})} - m_{1,-,1}\frac{\pi^{(1)}_{11}v_{11}}{u_{10}\pi^{(1)}_{01}v_{10} + +u_{11}\pi^{(1)}_{11}v_{11}} \\
&+ \frac{m_{-,-,0}}{p_{-,-,0}}\{(1-\pi^{(1)}_{00})(1-\pi^{(2)}_{00})(1-v_{00}) - (1-\pi^{(1)}_{10})(1-\pi^{(2)}_{10})(1-v_{11})\} \\
&+ \frac{m_{-,-,1}}{p_{-,-,1}}\{(1-\pi^{(1)}_{01})(1-\pi^{(2)}_{01})v_{00} - (1-\pi^{(1)}_{11})(1-\pi^{(2)}_{11})v_{11}\},
\end{aligned}$$

$$
\begin{aligned}
a_2^T\tilde{n} \;=\;& 0(n_{000}) + \frac{n_{010}}{u_{01}} + 0(n_{100}) - \frac{n_{110}}{u_{11}} + 0(n_{001}) + \frac{n_{011}}{u_{01}} + 0(n_{101}) - \frac{n_{111}}{u_{11}} \\
& +0(m_{-,0,0}) + 0(m_{-,0,1}) \\
& +m_{-,1,0}\frac{\pi_{00}^{(2)}(1-v_{01}) - \pi_{10}^{(2)}(1-v_{11})}{u_{01}\pi_{00}^{(2)}(1-v_{01}) + u_{11}\pi_{10}^{(2)}(1-v_{11})} + m_{-,1,1}\frac{\pi_{01}^{(2)}v_{01} - \pi_{11}^{(2)}v_{11}}{u_{01}\pi_{01}^{(2)}v_{01} + u_{11}\pi_{11}^{(2)}v_{11}} \\
& +m_{0,-,0}\frac{\pi_{10}^{(1)}(1-v_{01})}{u_{00}\pi_{00}^{(1)}(1-v_{00}) + u_{01}\pi_{10}^{(1)}(1-v_{01})} + m_{0,-,1}\frac{\pi_{11}^{(1)}v_{01}}{u_{00}\pi_{01}^{(1)}v_{00} + u_{01}\pi_{11}^{(1)}v_{01}} \\
& -m_{1,-,0}\frac{\pi_{10}^{(1)}(1-v_{11})}{u_{10}\pi_{00}^{(1)}(1-v_{10}) + u_{11}\pi_{10}^{(1)}(1-v_{11})} - m_{1,-,1}\frac{\pi_{11}^{(1)}v_{11}}{u_{10}\pi_{01}^{(1)}v_{10} + +u_{11}\pi_{11}^{(1)}v_{11}} \\
& +\frac{m_{-,-,0}}{p_{-,-,0}}\{(1-\pi_{10}^{(1)})(1-\pi_{00}^{(2)})(1-v_{01}) - (1-\pi_{10}^{(1)})(1-\pi_{10}^{(2)})(1-v_{11})\} \\
& +\frac{m_{-,-,1}}{p_{-,-,1}}\{(1-\pi_{11}^{(1)})(1-\pi_{01}^{(2)})v_{01} - (1-\pi_{11}^{(1)})(1-\pi_{11}^{(2)})v_{11}\},
\end{aligned}
$$

$$
\begin{aligned}
a_3^T\tilde{n} \;=\;& 0(n_{000}) + 0(n_{010}) + \frac{n_{100}}{u_{10}} - \frac{n_{110}}{u_{11}} + 0(n_{001}) + 0(n_{011}) + \frac{n_{101}}{u_{10}} - \frac{n_{111}}{u_{11}} \\
& +m_{-,0,0}\frac{\pi_{10}^{(2)}(1-v_{10})}{u_{00}\pi_{00}^{(2)}(1-v_{00}) + u_{10}\pi_{10}^{(2)}(1-v_{10})} + m_{-,0,1}\frac{\pi_{11}^{(2)}v_{10}}{u_{00}\pi_{01}^{(2)}v_{00} + u_{10}\pi_{11}^{(2)}v_{10}} \\
& -m_{-,1,0}\frac{\pi_{10}^{(2)}(1-v_{11})}{u_{01}\pi_{00}^{(2)}(1-v_{01}) + u_{11}\pi_{10}^{(2)}(1-v_{11})} - m_{-,1,1}\frac{\pi_{11}^{(2)}v_{11}}{u_{01}\pi_{01}^{(2)}v_{01} + u_{11}\pi_{11}^{(2)}v_{11}} \\
& +0(m_{0,-,0}) + 0(m_{0,-,1}) \\
& +m_{1,-,0}\frac{\pi_{00}^{(1)}(1-v_{10}) - \pi_{10}^{(1)}(1-v_{11})}{u_{10}\pi_{00}^{(1)}(1-v_{10}) + u_{11}\pi_{10}^{(1)}(1-v_{11})} + m_{1,-,1}\frac{\pi_{01}^{(1)}v_{10} - \pi_{11}^{(1)}v_{11}}{u_{10}\pi_{01}^{(1)}v_{10} + +u_{11}\pi_{11}^{(1)}v_{11}} \\
& +\frac{m_{-,-,0}}{p_{-,-,0}}\{(1-\pi_{00}^{(1)})(1-\pi_{10}^{(2)})(1-v_{10}) - (1-\pi_{10}^{(1)})(1-\pi_{10}^{(2)})(1-v_{11})\} \\
& +\frac{m_{-,-,1}}{p_{-,-,1}}\{(1-\pi_{01}^{(1)})(1-\pi_{11}^{(2)})v_{10} - (1-\pi_{11}^{(1)})(1-\pi_{11}^{(2)})v_{11}\},
\end{aligned}
$$

$$
\begin{aligned}
a_4^T\tilde{n} \;=\;& -\frac{n_{000}}{1-v_{00}} + 0(n_{010}) + 0(n_{100}) + 0(n_{110}) + \frac{n_{001}}{v_{00}} + 0(n_{011}) + 0(n_{101}) + 0(n_{111}) \\
& -m_{-,0,0}\frac{u_{00}\pi_{00}^{(2)}}{u_{00}\pi_{00}^{(2)}(1-v_{00}) + u_{10}\pi_{10}^{(2)}(1-v_{10})} + m_{-,0,1}\frac{u_{00}\pi_{01}^{(2)}}{u_{00}\pi_{01}^{(2)}v_{00} + u_{10}\pi_{11}^{(2)}v_{10}} \\
& +0(m_{-,1,0}) + 0(m_{-,1,1}) \\
& -m_{0,-,0}\frac{u_{00}\pi_{00}^{(1)}}{u_{00}\pi_{00}^{(1)}(1-v_{00}) + u_{01}\pi_{10}^{(1)}(1-v_{01})} + m_{0,-,1}\frac{u_{00}\pi_{01}^{(1)}}{u_{00}\pi_{01}^{(1)}v_{00} + u_{01}\pi_{11}^{(1)}v_{01}} \\
& +0(m_{1,-,0}) + 0(m_{1,-,1})
\end{aligned}
$$

$$-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{00}(1-\pi^{(1)}_{00})(1-\pi^{(2)}_{00})\}+\frac{m_{-,-,1}}{p_{-,-,1}}\{u_{00}(1-\pi^{(1)}_{01})(1-\pi^{(2)}_{01})\},$$

$$\begin{aligned}
a_5^T\tilde{n} \;=\;& 0(n_{000})-\frac{n_{010}}{1-v_{01}}+0(n_{100})+0(n_{110})+0(n_{001})+\frac{n_{011}}{v_{01}}+0(n_{101})+0(n_{111})\\
&+0(m_{-,0,0})+0(m_{-,0,1})\\
&-m_{-,1,0}\frac{u_{01}\pi^{(2)}_{00}}{u_{01}\pi^{(2)}_{00}(1-v_{01})+u_{11}\pi^{(2)}_{10}(1-v_{11})}+m_{-,1,1}\frac{u_{01}\pi^{(2)}_{01}}{u_{01}\pi^{(2)}_{01}v_{01}+u_{11}\pi^{(2)}_{11}v_{11}}\\
&-m_{0,-,0}\frac{u_{01}\pi^{(1)}_{10}}{u_{00}\pi^{(1)}_{00}(1-v_{00})+u_{01}\pi^{(1)}_{10}(1-v_{01})}+m_{0,-,1}\frac{u_{01}\pi^{(1)}_{11}}{u_{00}\pi^{(1)}_{01}v_{00}+u_{01}\pi^{(1)}_{11}v_{01}}\\
&+0(m_{1,-,0})+0(m_{1,-,1})\\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{01}(1-\pi^{(1)}_{10})(1-\pi^{(2)}_{00})\}+\frac{m_{-,-,1}}{p_{-,-,1}}\{u_{01}(1-\pi^{(1)}_{11})(1-\pi^{(2)}_{01})\},
\end{aligned}$$

$$\begin{aligned}
a_6^T\tilde{n} \;=\;& 0(n_{000})+0(n_{010})-\frac{n_{100}}{1-v_{10}}+0(n_{110})+0(n_{001})+0(n_{011})+\frac{n_{101}}{v_{10}}+0(n_{111})\\
&-m_{-,0,0}\frac{u_{10}\pi^{(2)}_{10}}{u_{00}\pi^{(2)}_{00}(1-v_{00})+u_{10}\pi^{(2)}_{10}(1-v_{10})}+m_{-,0,1}\frac{u_{10}\pi^{(2)}_{11}}{u_{00}\pi^{(2)}_{01}v_{00}+u_{10}\pi^{(2)}_{11}v_{10}}\\
&+0(m_{-,1,0})+0(m_{-,1,1})\\
&+0(m_{0,-,0})+0(m_{0,-,1})\\
&-m_{1,-,0}\frac{u_{10}\pi^{(1)}_{00}}{u_{10}\pi^{(1)}_{00}(1-v_{10})+u_{11}\pi^{(1)}_{10}(1-v_{11})}+m_{1,-,1}\frac{u_{10}\pi^{(1)}_{01}}{u_{10}\pi^{(1)}_{01}v_{10}++u_{11}\pi^{(1)}_{11}v_{11}}\\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{10}(1-\pi^{(1)}_{00})(1-\pi^{(2)}_{10})\}+\frac{m_{-,-,1}}{p_{-,-,1}}\{u_{10}(1-\pi^{(1)}_{01})(1-\pi^{(2)}_{11})\},
\end{aligned}$$

$$\begin{aligned}
a_7^T\tilde{n} \;=\;& 0(n_{000})+0(n_{010})+0(n_{100})-\frac{n_{110}}{1-v_{11}}+0(n_{001})+0(n_{011})+0(n_{101})+\frac{n_{111}}{v_{11}}\\
&+0(m_{-,0,0})+0(m_{-,0,1})\\
&-m_{-,1,0}\frac{u_{11}\pi^{(2)}_{10}}{u_{01}\pi^{(2)}_{00}(1-v_{01})+u_{11}\pi^{(2)}_{10}(1-v_{11})}+m_{-,1,1}\frac{u_{11}\pi^{(2)}_{11}}{u_{01}\pi^{(2)}_{01}v_{01}+u_{11}\pi^{(2)}_{11}v_{11}}\\
&+0(m_{0,-,0})+0(m_{0,-,1})\\
&-m_{1,-,0}\frac{u_{11}\pi^{(1)}_{10}}{u_{10}\pi^{(1)}_{00}(1-v_{10})+u_{11}\pi^{(1)}_{10}(1-v_{11})}+m_{1,-,1}\frac{u_{11}\pi^{(1)}_{11}}{u_{10}\pi^{(1)}_{01}v_{10}++u_{11}\pi^{(1)}_{11}v_{11}}\\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{11}(1-\pi^{(1)}_{10})(1-\pi^{(2)}_{10})\}+\frac{m_{-,-,1}}{p_{-,-,1}}\{u_{11}(1-\pi^{(1)}_{11})(1-\pi^{(2)}_{11})\},
\end{aligned}$$

$$a_8^T\tilde{n} \;=\; \frac{n_{000}}{\pi^{(1)}_{00}}+0(n_{010})+\frac{n_{100}}{\pi^{(1)}_{00}}+0(n_{110})+0(n_{001})+0(n_{011})+0(n_{101})+0(n_{111})$$

$$
\begin{aligned}
&-\frac{m_{-,0,0}}{(1-\pi_{00}^{(1)})} + 0(m_{-,0,1}) + 0(m_{-,1,0}) + 0(m_{-,1,1}) \\
&+m_{0,-,0}\frac{u_{00}(1-v_{00})}{u_{00}\pi_{00}^{(1)}(1-v_{00}) + u_{01}\pi_{10}^{(1)}(1-v_{01})} + 0(m_{0,-,1}) \\
&-m_{1,-,0}\frac{u_{10}(1-v_{11})}{u_{10}\pi_{00}^{(1)}(1-v_{10}) + u_{11}\pi_{10}^{(1)}(1-v_{11})} + 0(m_{1,-,1}) \\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{00}\pi_{00}^{(1)}(1-\pi_{00}^{(2)})(1-v_{00}) + u_{10}(1-\pi_{10}^{(2)})(1-v_{10})\} + 0(m_{-,-,1}),
\end{aligned}
$$

$$
\begin{aligned}
a_9^T\tilde{n} \;=\;& 0(n_{000}) + 0(n_{010}) + 0(n_{100}) + 0(n_{110}) + \frac{n_{001}}{\pi_{01}^{(1)}} + 0(n_{011}) + \frac{n_{101}}{\pi_{01}^{(1)}} + 0(n_{111}) \\
&+0(m_{-,0,0}) - \frac{m_{-,0,1}}{1-\pi_{01}^{(1)}} + 0(m_{-,1,0}) + 0(m_{-,1,1}) \\
&+0(m_{0,-,0}) + m_{0,-,1}\frac{u_{00}v_{00}}{u_{00}\pi_{01}^{(1)}v_{00} + u_{01}\pi_{11}^{(1)}v_{01}} \\
&+0(m_{1,-,0}) + m_{1,-,1}\frac{u_{10}u_{10}}{u_{10}\pi_{01}^{(1)}v_{10} + +u_{11}\pi_{11}^{(1)}v_{11}} \\
&+0(m_{-,-,0}) - \frac{m_{-,-,1}}{p_{-,-,1}}\{u_{00}(1-\pi_{01}^{(2)})v_{00}u_{10}(1-\pi_{11}^{(2)})v_{10}\},
\end{aligned}
$$

$$
\begin{aligned}
a_{10}^T\tilde{n} \;=\;& 0(n_{000}) + \frac{n_{010}}{\pi_{10}^{(1)}} + 0(n_{100}) + \frac{n_{110}}{\pi_{10}^{(1)}} + 0(n_{001}) + 0(n_{011}) + 0(n_{101}) + 0(n_{111}) \\
&+0(m_{-,0,0}) + 0(m_{-,0,1}) - \frac{m_{-,1,0}}{1-\pi_{10}^{(1)}} + 0(m_{-,1,1}) \\
&+m_{0,-,0}\frac{u_{01}(1-v_{01})}{u_{00}\pi_{00}^{(1)}(1-v_{00}) + u_{01}\pi_{10}^{(1)}(1-v_{01})} + 0(m_{0,-,1}) \\
&+m_{1,-,0}\frac{u_{10}(1-v_{10})}{u_{10}\pi_{00}^{(1)}(1-v_{10}) + u_{11}\pi_{10}^{(1)}(1-v_{11})} + 0(m_{1,-,1}) \\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{01}(1-\pi_{00}^{(2)})(1-v_{01}) + u_{11}(1-\pi_{10}^{(2)})(1-v_{11})\} + 0(m_{-,-,1}),
\end{aligned}
$$

$$
\begin{aligned}
a_{11}^T\tilde{n} \;=\;& 0(n_{000}) + 0(n_{010}) + 0(n_{100}) - 0(n_{110}) + 0(n_{001}) + \frac{n_{011}}{\pi_{11}^{(1)}} + 0(n_{101}) + \frac{n_{111}}{\pi_{11}^{(1)}} \\
&+0(m_{-,0,0}) + 0(m_{-,0,1}) + 0(m_{-,1,0}) - \frac{m_{-,1,1}}{(1-\pi_{11}^{(1)})} \\
&+0(m_{0,-,0}) + m_{0,-,1}\frac{u_{01}v_{01}}{u_{00}\pi_{01}^{(1)}v_{00} + u_{01}\pi_{11}^{(1)}v_{01}} \\
&+0(m_{1,-,0}) + m_{1,-,1}\frac{u_{11}v_{11}}{u_{10}\pi_{01}^{(1)}v_{10} + +u_{11}\pi_{11}^{(1)}v_{11}}
\end{aligned}
$$

$$+0(m_{-,-,0}) - \frac{m_{-,-,1}}{p_{-,-,1}}\{u_{01}(1-\pi_{01}^{(2)})v_{01} + u_{11}(1-\pi_{11}^{(2)})v_{11}\},$$

$$\begin{aligned}
a_{12}^T\tilde{n} &= \frac{n_{000}}{\pi_{00}^{(2)}} + \frac{n_{010}}{\pi_{00}^{(2)}} + 0(n_{100}) + \frac{n_{110}}{u_{11}} + 0(n_{001}) + 0(n_{011}) + 0(n_{101}) + 0(n_{111}) \\
&+m_{-,0,0}\frac{u_{00}(1-v_{00})}{u_{00}\pi_{00}^{(2)}(1-v_{00}) + u_{10}\pi_{10}^{(2)}(1-v_{10})} + 0(m_{-,0,1}) \\
&+m_{-,1,0}\frac{u_{01}(1-v_{01})}{u_{01}\pi_{00}^{(2)}(1-v_{01}) + u_{11}\pi_{10}^{(2)}(1-v_{11})} + 0(m_{-,1,1}) \\
&-\frac{m_{0,-,0}}{1-\pi_{00}^{(2)}} + 0(m_{0,-,1}) + 0(m_{1,-,0}) + 0(m_{1,-,1}) \\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{00}(1-\pi_{00}^{(1)})(1-v_{00}) - u_{01}(1-\pi_{10}^{(1)})(1-v_{01})\} + 0(m_{-,-,1}),
\end{aligned}$$

$$\begin{aligned}
a_{13}^T\tilde{n} &= 0(n_{000}) + 0(n_{010}) + 0(n_{100}) + 0(n_{110}) + \frac{n_{001}}{\pi_{01}^{(2)}} + \frac{n_{011}}{\pi_{01}^{(2)}} + 0(n_{101}) + 0(n_{111}) \\
&+0(m_{-,0,0}) + m_{-,0,1}\frac{u_{00}v_{00}}{u_{00}\pi_{01}^{(2)}v_{00} + u_{10}\pi_{11}^{(2)}v_{10}} \\
&+0(m_{-,1,0}) + m_{-,1,1}\frac{u_{01}v_{01}}{u_{01}\pi_{01}^{(2)}v_{01} + u_{11}\pi_{11}^{(2)}v_{11}} \\
&+0(m_{0,-,0}) - \frac{m_{0,-,1}}{1-\pi_{01}^{(2)}} + 0(m_{1,-,0}) + 0(m_{1,-,1}) \\
&+0(m_{-,-,0}) + \frac{m_{-,-,1}}{p_{-,-,1}}\{u_{00}(1-\pi_{01}^{(1)})v_{00} + u_{01}(1-\pi_{11}^{(1)})v_{01}\},
\end{aligned}$$

$$\begin{aligned}
a_{14}^T\tilde{n} &= 0(n_{000}) + 0(n_{010}) + \frac{n_{100}}{\pi_{10}^{(2)}} + \frac{n_{110}}{\pi_{10}^{(2)}} + 0(n_{001}) + 0(n_{011}) + 0(n_{101}) + 0(n_{111}) \\
&+m_{-,0,0}\frac{u_{10}(1-v_{10})}{u_{00}\pi_{00}^{(2)}(1-v_{00}) + u_{10}\pi_{10}^{(2)}(1-v_{10})} + 0(m_{-,0,1}) \\
&+m_{-,1,0}\frac{u_{11}(1-v_{11})}{u_{01}\pi_{00}^{(2)}(1-v_{01}) + u_{11}\pi_{10}^{(2)}(1-v_{11})} + 0(m_{-,1,1}) \\
&-\frac{m_{0,-,0}}{1-\pi_{10}^{(2)}} + 0(m_{0,-,1}) - \frac{m_{1,-,0}}{1-\pi_{10}^{(2)}} + 0(m_{1,-,1}) \\
&-\frac{m_{-,-,0}}{p_{-,-,0}}\{u_{10}(1-\pi_{00}^{(1)})(1-v_{10}) + u_{11}(1-\pi_{10}^{(1)})(1-v_{11})\} + 0(m_{-,-,1}),
\end{aligned}$$

$$\begin{aligned}
a_{15}^T\tilde{n} &= 0(n_{000}) + 0(n_{010}) + 0(n_{100}) + 0(n_{110}) + 0(n_{001}) + 0(n_{011}) + \frac{n_{101}}{\pi_{11}^{(2)}} + \frac{n_{111}}{\pi_{11}^{(2)}} \\
&+0(m_{-,0,0}) + m_{-,0,1}\frac{u_{10}v_{10}}{u_{00}\pi_{01}^{(2)}v_{00} + u_{10}\pi_{11}^{(2)}v_{10}}
\end{aligned}$$

$$+0(m_{-,1,0}) + m_{-,1,1}\frac{u_{11}v_{11}}{u_{01}\pi_{01}^{(2)}v_{01} + u_{11}\pi_{11}^{(2)}v_{11}}$$
$$+0(m_{0,-,0}) + 0(m_{0,-,1}) + 0(m_{1,-,0}) - \frac{m_{1,-,1}}{1-\pi_{11}^{(2)}}$$
$$+0(m_{-,-,0}) - \frac{m_{-,-,1}}{p_{-,-,1}}\{u_{10}(1-\pi_{01}^{(1)})v_{10} - u_{11}(1-\pi_{11}^{(1)})v_{11}\}.$$

A close inspection shows that the 15 rows of A are linearly independent, completing the proof of the fact that the rank of A is 15.

Lemma 1. *Let A be an $m_1 \times m_2$ matrix with full row rank (i.e., $r_A = m_1$), and $m_2 > m_1$, and Ω be an $m_2 \times m_2$ symmetric positive semidefinite matrix and $r_\Omega = r(\Omega) > r(A) = m_1$. Then $r(A\Omega A^T) = m_1$.*

Proof: Since Ω is a positive semidefinite matrix, we can write $\Omega = C\Lambda C^T$ for a nonsingular matrix C and a diagonal matrix Λ such that

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where Λ_1 is an $r_\Omega \times r_\Omega$ diagonal matrix with all diagonal elements positive. Then $A\Omega A^T = AC\Lambda C^T A^T = B\Lambda B$, where $B = AC$. Observe that $r(B) = r(AC) = r(A)$ as C is a nonsingular matrix. Since B is an $m_1 \times m_2$ matrix of rank m_1, there exists a nonsingular matrix Q of the order $m_2 \times m_2$ such that $BQ = (I_{m_1} : 0)$, where I_{m_1} is an identity matrix of the order of m_1. Now, let W_{11} be a nonsingular $r_\Omega \times r_\Omega$ matrix such that

$$Q^{-1}\Lambda Q^{-T} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

Then it is easy to see that $r(B\Lambda B^T) = r(BQQ^{-1}\Lambda Q^{-T}Q^T B^T) = m_1 = r(A)$.

W-A3 Regularity conditions:

C1. $\mathrm{pr}(R_1 = \cdots = R_p = 1|X, Y, Z) > 0$ for all (X, Y, Z) with probability 1.

C2. Conditional on X, Y, Z, $R_1, \ldots, R_p$ are independent.

C3. Missingness of X_j does not depend on X_j itself, for $j = 1, \ldots, p$.

C4. $\mathrm{pr}(R_k = 1|X, Y, Z)$ is twice continuously differentiable function of α_k for $k = 1, \ldots, p$.

C5. $\int h(Y, X, Z, \omega) dP(X_r|X_{-(r)}, Z, R_1 = \cdots = R_p = 1)$ is bounded away from 0 for all (X, Y, Z).

C6. $f(Y|X, Z, \beta)$ is a twice continuously differentiable function of β.

C7. D is non-singular in an open neighborhood around the true value of θ.

C8. The parameter space of θ is a compact subset of an Euclidean space.

If X, Y, Z are discrete variables with finite many possible values, then C1 is easy to verify from a given data set. Conditions C2 and C3 are not verifiable without relevant external source of information. When one suspects that the MAR assumption is inadequate, the NI- assumption may be adopted.

W-A4 Components of matrix D

With simplified notations $X = (X_1, X_2)^T$, $S_\beta = \partial\log\{f(Y|X,Z)\}/\partial\beta$, $\pi_k = 1 - \bar{\pi}_k = \mathrm{pr}(R_k = 1|Y, X_{(-k)}, Z)$, $S_{\alpha_k,1} = \partial\log(\pi_k)/\alpha_k$, $S_{\alpha_k,0} = \partial\log(\bar{\pi}_k)/\alpha_k$, for $k = 1, 2$, it is easy to see that $(1/n)\partial S_{\widehat{f}_x,\alpha_k}/\partial\alpha_k \xrightarrow{P} A_{\alpha_k\alpha_k}$, $(1/n)\partial S_{\widehat{f}_x,\beta}/\partial\beta \xrightarrow{P} A_{\beta\beta}$, $(1/n)\partial S_{\widehat{f}_x,\beta}/\partial\alpha_k \xrightarrow{P} A_{\beta\alpha_k}$, $(1/n)\partial S_{\widehat{f}_x,\alpha_k}/\partial\beta \xrightarrow{P} A_{\alpha_k\beta}$. Moreover,

$$
\begin{aligned}
A_{\alpha_1\alpha_1} &= E\left(\pi_1\pi_2\frac{\partial}{\partial\alpha_1}S_{\alpha_1,1} + \bar{\pi}_1\pi_2\frac{\partial}{\partial\alpha_1}S_{\alpha_1,0}\right)\\
&\quad +E\left\{\pi_1\bar{\pi}_2\left(E\left(\frac{\partial}{\partial\alpha_1}S_{\alpha_1,1}|Y, X_1, Z, R_1 = 1, R_2 = 0\right)+\mathrm{cov}\left(S_{\alpha_1,1}, S^T_{\alpha_1,1}|Y, X_1, Z, R_1 = 1, R_2 = 0\right)\right.\right.\\
&\quad \left.\left.-\mathrm{cov}\left[S_{\alpha_1,1}, \frac{\partial}{\partial\alpha_1}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, X_1, Z, R_1 = 1, R_2 = 0\right]\right)\right\}\\
&\quad +E\left\{\bar{\pi}_1\bar{\pi}_2\left(E\left(\frac{\partial}{\partial\alpha_1}S_{\alpha_1,0}|Y, Z, R_1 = R_2 = 0\right)+\mathrm{cov}\left(S_{\alpha_1,0}, S^T_{\alpha_1,0}|Y, Z, R_1 = R_2 = 0\right)\right.\right.\\
&\quad \left.\left.-\mathrm{cov}\left[S_{\alpha_1,0}, \frac{\partial}{\partial\alpha_1}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, Z, R_1 = R_2 = 0\right]\right)\right\},\\
A_{\alpha_2\alpha_2} &= E\left(\pi_1\pi_2\frac{\partial}{\partial\alpha_2}S_{\alpha_2,1} + \pi_1\bar{\pi}_2\frac{\partial}{\partial\alpha_2}S_{\alpha_2,0}\right)\\
&\quad +E\left\{\bar{\pi}_1\pi_2\left(E\left[\frac{\partial}{\partial\alpha_2}S_{\alpha_2,1}|Y, X_2, Z, R_1 = 0, R_2 = 1\right)+\mathrm{cov}\left(S_{\alpha_2,1}, S^T_{\alpha_2,1}|Y, X_2, Z, R_1 = 0, R_2 = 1\right)\right.\right.\\
&\quad \left.\left.-\mathrm{cov}\left[S_{\alpha_2,1}, \frac{\partial}{\partial\alpha_2}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, X_2, Z, R_1 = 0, R_2 = 1\right]\right)\right\}\\
&\quad +E\left\{\bar{\pi}_1\bar{\pi}_2\left(E\left[\frac{\partial}{\partial\alpha_2}S_{\alpha_2,0}|Y, Z, R_1 = 0, R_2 = 0\right)+\mathrm{cov}\left(S_{\alpha_2,0}, S^T_{\alpha_2,0}|Y, Z, R_1 = 0, R_2 = 0\right)\right.\right.\\
&\quad \left.\left.-\mathrm{cov}\left[S_{\alpha_2,0}, \frac{\partial}{\partial\alpha_2}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, Z, R_1 = 0, R_2 = 0\right]\right)\right\},\\
A_{\alpha_1\alpha_2} &= -E\left(\pi_{1,1}\bar{\pi}_{1,2}\mathrm{cov}\left[S_{\alpha_1,0}, \frac{\partial}{\partial\alpha_2}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, X_1, Z, R_1 = 1, R_2 = 0\right]\right)\\
&\quad E\left\{\bar{\pi}_1\bar{\pi}_2\left(\mathrm{cov}\left\{S_{\alpha_1,0}, S_{\alpha_2,0}|Y, Z, R_1 = R_2 = 0\right\}\right.\right.\\
&\quad \left.\left.-\mathrm{cov}\left[S_{\alpha_1,0}, \frac{\partial}{\partial\alpha_2}\log\{\mathrm{pr}(R_1 = R_2 = 1|X, Z)\}|Y, Z, R_1 = R_2 = 0\right]\right)\right\},\\
A_{\beta\beta} &= E\left\{\pi_1\pi_2\frac{\partial}{\partial\beta}S_\beta\right\}+E\left\{\pi_1\bar{\pi}_2\left(E\left[\frac{\partial}{\partial\beta}S_\beta + S_\beta S^T_\beta|Y, X_1, Z, R_1 = 1, R_2 = 0\right]\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\text{cov}\left[S_\beta, \frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}|Y,X_1,Z,R_1=1,R_2=0\right]\bigg)\bigg\}\\
&+E\bigg\{\pi_1\pi_2\bigg(E\left[\frac{\partial}{\partial\beta}S_{1,\beta}+S_\beta S_\beta^T|Y,X_2,Z,R_1=0,R_2=1\right]\\
&-\text{cov}\left[S_\beta, \frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}Y,X_2,Z,R_1=0,R_2=1\right]\bigg)\bigg\}\\
&+E\bigg\{\bar{\pi}_1\bar{\pi}_2\bigg(E\left[\frac{\partial}{\partial\beta}S_\beta+S_\beta S_{1,\beta}^T|Y,Z,R_1=R_2=0\right]\\
&-\text{cov}\left[S_\beta, \frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}|Y,Z,R_1=R_2=0\right]\bigg)\bigg\},\\
A_{\beta\alpha_1}=&E\bigg\{\pi_1\bar{\pi}_2\bigg(\text{cov}(S_\beta,S_{\alpha_1,1}|Y,X_1,Z,R_1=1,R_2=0)-\text{cov}[S_\beta,\frac{\partial}{\partial\alpha_1}\log\{\text{pr}(R_1=R_2=1|X,Z)\}\\
&|Y,X_1,Z,R_1=1,R_2=0]\bigg)\bigg\}-E\bigg(\bar{\pi}_1\pi_2\text{cov}[S_\beta,\frac{\partial}{\partial\alpha_1}\log\{\text{pr}(R_1=R_2=1|X,Z)\}\\
&|Y,X_2,Z,R_1=0,R_2=1]\bigg)+E\bigg\{\bar{\pi}_1\bar{\pi}_2\bigg(\text{cov}(S_\beta,S_{\alpha_1,0}|Y,Z,R_1=R_2=0)\\
&-\text{cov}[S_\beta,\frac{\partial}{\partial\alpha_1}\log\{\text{pr}(R_1=R_2=1|X,Z)\}|Y,Z,R_1=0,R_2=0]\bigg)\bigg\},\\
A_{\alpha_1\beta}=&E\bigg\{\pi_1\bar{\pi}_2\bigg(\text{cov}(S_\beta,S_{\alpha_1,1}|Y,X_1,Z,R_1=1,R_2=0)-\text{cov}[S_{\alpha_1,1},\frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}\\
&|Y,X_1,Z,R_1=1,R_2=0]\bigg)\bigg\}+E\bigg\{\bar{\pi}_1\pi_2\bigg(\text{cov}(S_\beta,S_{\alpha_1,0}|Y,X_2,Z,R_1=0,R_2=1)\\
&-\text{cov}[S_{\alpha_1,0},\frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}|Y,X_2,Z,R_1=0,R_2=1]\bigg)\bigg\}\\
&+E\bigg\{\bar{\pi}_1\bar{\pi}_2\bigg(\text{cov}(S_\beta,S_{\alpha_1,0}|Y,Z,R_1=R_2=0)\\
&-\text{cov}[S_{\alpha_1,0},\frac{\partial}{\partial\beta}\log\{\text{pr}(R_1=R_2=1|X,Z)\}|Y,Z,R_1=0,R_2=0]\bigg)\bigg\}.
\end{aligned}
$$

Likewise, $A_{\beta\alpha_2}$ and $A_{\alpha_2\beta}$ can be expressed in the same fashion as $A_{\beta\alpha_1}$ and $A_{\alpha_1\beta}$, respectively.

W-A5 Proof of Theorem 1.

In order to obtain the influence function representation of the $\widehat{\theta}$ we write the estimating equations asymptotically as a sum of n independent terms. The second term of $S_{\widehat{f}_x,\beta}$ is

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^n R_{i1}(1-R_{i2})\frac{\int S_\beta(Y_i,X_i,Z_i)h(Y_i,X_i,Z_i,\pi_{i1})\widehat{f}(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}{\int h(Y_i,X_i,Z_i,\pi_{i1})\widehat{f}(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^n R_{i1}(1-R_{i2})\frac{\sum_{j=1}^n S_\beta(Y_i,X_i,Z_i)h(Y_i,X_i,Z_i,\pi_{i1})I(X_{j1}=X_{i1},Z_j=Z_i,R_{j1}=R_{j2}=1)}{\sum_{j=1}^n h(Y_i,X_i,Z_i,\pi_{i1})I(X_{j1}=X_{i1},Z_j=Z_i,R_{j1}=R_{j2}=1)}.
\end{aligned}
$$

Now, using the Hadamard differentiability of $\int S_\beta(Y,X,Z)h(Y,X,Z,\pi_1)dP(X_2|X_1,Z,R_1=R_2=1)$ and $\int h(Y,X,Z,\pi_1)dP(X_2|X_1,Z,R_1=R_2=1)$ we can write

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^n R_{i1}(1-R_{i2})\bigg\{\frac{\int S_\beta(Y_i,X_i,Z_i)h(Y_i,X_i,Z_i,\pi_{i1})\widehat{f}(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}{\int h(Y_i,X_i,Z_i,\pi_{i1})\widehat{f}(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}\\
&\quad-\frac{\int S_\beta(Y_i,X_i,Z_i)h(Y_i,X_i,Z_i,\pi_{i1})f(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}{\int h(Y_i,X_i,Z_i,\pi_{i1})f(X_{i2}|X_{i(-2)},Z_i,R_{i1}=R_{i2}=1)}\bigg\}\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^n R_{i1}(1-R_{i2})\frac{1}{n}\sum_{j=1}^n\frac{h(Y_i,X_i,Z_i,\pi_{i1})}{a(Y_i,X_{i1},Z_i,\pi_{i1})}\bigg\{S_\beta(Y_i,X_i,Z_i)-\frac{a_\beta(Y_i,X_{i1},Z_i,\pi_{i1})}{a(Y_i,X_{i1},Z_i,\pi_{i1})}\bigg\}\\
&\quad\times\frac{I(X_{j1}=X_{i1},Z_j=Z_i,R_{j1}=R_{j2}=1)}{\mathrm{pr}(X_{i1},Z_i,R_1=R_2=1)}+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^n R_{i1}(1-R_{i2})\frac{1}{n}\sum_{j=1}^n R_{j1}R_{j2}\frac{h(Y_i,X_i,Z_i,\pi_{i1})}{a(Y_i,X_{i1},Z_i,\pi_{i1})}\bigg\{S_\beta(Y_i,X_i,Z_i)-\frac{a_\beta(Y_i,X_{i1},Z_i,\pi_{i1})}{a(Y_i,X_{i1},Z_i,\pi_{i1})}\bigg\}\\
&\quad\times\frac{I(X_{j1}=X_{i1},Z_j=Z_i)}{\mathrm{pr}(R_1=R_2=1|X_i,Z_i)\mathrm{pr}(X_{i1},Z_i)}+o_p(1).
\end{aligned}
$$

After interchanging the order of the sums and then applying the strong law of large numbers we can write the above dominating term as $n^{-1/2}\sum_{j=1}^n \Upsilon_{j,\beta,10}$. Following the same technique, we linearize the other two terms of $S_{\widehat{f}_x,\beta}$ and consequently we write $n^{-1/2}S_{\widehat{f}_x,\beta}=n^{-1/2}\sum_{i=1}^n S^{\mathrm{adj}}_{i,\widehat{f},\beta}+o_p(1)$. Applying the same principles we linearize $S_{\widehat{f}_x,\alpha_k}$, $k=1,2$. Then the final conclusion follows from an application of Taylor's expansion of the estimating equations about the true parameters.

Table W-1: Results of the simulation study for scenario 1 based on 500 replications. Here the missingness mechanisms are non-ignorable. FD, CC, SP, EMP.SE, and EST.SE stand for the full data analysis, complete case, the proposed semiparametric method, empirical standard error, and estimated standard error, respectively.

Method		β_0	β_1	β_2	β_3	β_4
FD	Bias	−0.006	0.003	−0.003	0.003	−0.003
	EMP.SE	0.144	0.131	0.182	0.184	0.272
	EST.SE	0.147	0.132	0.182	0.189	0.265
	CP	0.968	0.952	0.94	0.954	0.938
	MSE	0.021	0.017	0.033	0.034	0.074
	$\text{logit}\{\text{pr}(R_k = 1 \mid X_1, X_2, Y, Z)\} = 0.25Y + 0.25Z + X_2 + X_1$					
CC	Bias	0.209	−0.007	−0.099	−0.103	0.035
	EMP.SE	0.259	0.173	0.292	0.298	0.386
	EST.SE	0.252	0.174	0.285	0.292	0.367
	CP	0.860	0.952	0.928	0.938	0.928
	MSE	0.111	0.029	0.095	0.099	0.149
Mean-score extension	Bias	0.096	−0.002	−0.097	−0.068	0.042
	EMP.SE	0.224	0.133	0.278	0.287	0.390
	EST.SE	0.226	0.135	0.277	0.286	0.374
	CP	0.913	0.969	0.937	0.946	0.924
	MSE	0.059	0.017	0.086	0.086	0.153
SP	Bias	0.056	0.001	−0.066	−0.035	0.028
	EMP.SE	0.216	0.133	0.272	0.288	0.385
	EST.SE	0.220	0.129	0.281	0.285	0.381
	CP	0.946	0.950	0.950	0.952	0.944
	MSE	0.049	0.017	0.078	0.079	0.148
	$\text{logit}\{\text{pr}(R_k = 1 \mid X_1, X_2, Y, Z)\} = 0.75 + Y + 0.25Z - X_1 + X_2$					
CC	Bias	0.449	−0.065	0.403	−0.241	0.178
	EMP.SE	0.189	0.175	0.280	0.222	0.372
	EST.SE	0.189	0.172	0.284	0.223	0.362
	CP	0.334	0.936	0.714	0.816	0.916
	MSE	0.237	0.034	0.241	0.108	0.169
Mean-score extension	Bias	0.044	−0.013	0.364	−0.173	−0.194
	EMP.SE	0.155	0.135	0.272	0.204	0.374
	EST.SE	0.161	0.137	0.279	0.210	0.370
	CP	0.958	0.954	0.756	0.892	0.918
	MSE	0.025	0.018	0.207	0.072	0.177
SP	Bias	0.023	−0.006	0.276	−0.116	−0.163
	EMP.SE	0.153	0.135	0.269	0.203	0.371
	EST.SE	0.165	0.133	0.295	0.213	0.387
	CP	0.972	0.954	0.874	0.940	0.938
	MSE	0.024	0.018	0.148	0.054	0.164