# Supplement to "Consistent Estimator for Logistic Mixed Effect Models"


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## Additional Simulation Results

Table S.1: Simulation results when random effect and covariates are independent. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim 0.8\text{N}(3,1) + 0.2\text{N}(6,1.5)$ and centered | | | | $X_{ij} \sim \text{N}(0.5,1)$ $Z_i \sim \text{Bernoulli}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 5.17 | 8.37 | 7.46 | 95.0% | 0.54 | 1.06 | 1.03 | 95.0% |
| $\widehat{\beta}_2$ | 9.92 | 10.40 | 8.84 | 94.9% | 1.30 | 1.20 | 1.18 | 95.5% |
| $\widehat{\beta}_3$ | −5.03 | 9.13 | 7.63 | 94.6% | −0.28 | 1.06 | 1.05 | 94.8% |
| | $R_i \sim 0.8\text{N}(3,1) + 0.2\text{N}(6,1.5)$ and centered | | | | $X_{ij} \sim \text{N}(0.5,1)$ $Z_i \sim \text{Poisson}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.92 | 9.07 | 7.31 | 96.0% | 0.49 | 0.93 | 0.96 | 96.6% |
| $\widehat{\beta}_2$ | 9.56 | 10.48 | 8.57 | 93.6% | 0.88 | 1.08 | 1.1 | 94.1% |
| $\widehat{\beta}_3$ | −5.80 | 8.56 | 7.35 | 94.8% | −0.45 | 1.02 | 0.99 | 94.7% |
| | $R_i \sim 0.8\text{N}(3,1) + 0.2\text{N}(6,1.5)$ and centered | | | | $X_{ij} \sim \text{N}(0.5,1)$ $Z_i \sim \text{Geometric}(0.7)$ | | | |
| $\widehat{\beta}_1$ | 3.72 | 7.53 | 6.73 | 95.7% | 0.90 | 0.93 | 0.96 | 95.2% |
| $\widehat{\beta}_2$ | 7.05 | 9.41 | 7.85 | 93.4% | 1.21 | 1.09 | 1.09 | 96.0% |
| $\widehat{\beta}_3$ | −4.74 | 8.32 | 6.83 | 94.2% | −0.37 | 0.97 | 0.97 | 94.9% |

Table S.2: Simulation results when random effect and covariates are independent. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim$ Gamma$(1, 1.25)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Bernoulli$(0.5)$ | | | |
| $\widehat{\beta}_1$ | 5.81 | 9.16 | 7.35 | 93.8% | −2.68 | 0.91 | 0.91 | 93.3% |
| $\widehat{\beta}_2$ | 8.36 | 11.19 | 8.45 | 94.3% | −2.12 | 1.02 | 1.04 | 94.4% |
| $\widehat{\beta}_3$ | −6.12 | 11.20 | 7.62 | 94.2% | −3.40 | 0.90 | 0.94 | 94.9% |
| | $R_i \sim$ Gamma$(1, 1.25)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Poisson$(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.86 | 9.02 | 7.52 | 93.7% | −2.11 | 0.88 | 0.90 | 94.8% |
| $\widehat{\beta}_2$ | 9.22 | 11.17 | 9.16 | 94.5% | −1.37 | 0.95 | 1.03 | 94.8% |
| $\widehat{\beta}_3$ | −7.08 | 10.33 | 8.01 | 92.8% | −2.99 | 0.92 | 0.93 | 95.4% |
| | $R_i \sim$ Gamma$(1, 1.25)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Geometric$(0.7)$ | | | |
| $\widehat{\beta}_1$ | 4.17 | 7.21 | 6.32 | 94.5% | −2.48 | 0.87 | 0.90 | 94.0% |
| $\widehat{\beta}_2$ | 7.54 | 10.04 | 7.63 | 93.9% | −1.93 | 0.95 | 1.02 | 95.4% |
| $\widehat{\beta}_3$ | −3.72 | 8.20 | 6.48 | 93.2% | −2.68 | 0.92 | 0.92 | 94.4% |

Table S.3: Simulation results when random effect and covariates are independent. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim$ t$(3)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Bernoulli$(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.37 | 8.90 | 8.03 | 95.3% | −0.33 | 1.21 | 1.21 | 95.0% |
| $\widehat{\beta}_2$ | 9.76 | 11.79 | 9.60 | 93.6% | −0.40 | 1.38 | 1.36 | 94.9% |
| $\widehat{\beta}_3$ | −6.51 | 9.66 | 8.39 | 94.4% | −1.52 | 1.20 | 1.24 | 95.9% |
| | $R_i \sim$ t$(3)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Poisson$(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.85 | 10.21 | 8.28 | 95.6% | −0.62 | 1.12 | 1.12 | 94.3% |
| $\widehat{\beta}_2$ | 8.30 | 13.49 | 9.72 | 94.1% | −0.23 | 1.18 | 1.26 | 95.2% |
| $\widehat{\beta}_3$ | −7.22 | 10.94 | 8.79 | 94.5% | −1.91 | 1.11 | 1.15 | 95.4% |
| | $R_i \sim$ t$(3)$ and centered | | | | $X_{ij} \sim$ N$(0.5, 1)$ $Z_i \sim$ Geometric$(0.7)$ | | | |
| $\widehat{\beta}_1$ | 3.80 | 9.02 | 7.74 | 95.8% | −0.03 | 1.05 | 1.13 | 95.4% |
| $\widehat{\beta}_2$ | 7.63 | 10.71 | 9.16 | 94.9% | 1.14 | 1.31 | 1.28 | 94.7% |
| $\widehat{\beta}_3$ | −4.96 | 9.37 | 8.03 | 94.1% | −1.06 | 1.16 | 1.15 | 95.1 % |

Table S.4: Simulation results when random effect and covariates are independent. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim \mathrm{N}(0,1)$ | | | | $X_{ij} \sim \mathrm{N}(0.5,1)$ $Z_i \sim$ Bernoulli(0.5) | | | |
| $\widehat{\beta}_1$ | 5.35 | 11.05 | 9.80 | 94.6% | 0.58 | 1.05 | 1.11 | 96.0% |
| $\widehat{\beta}_2$ | 10.48 | 15.19 | 11.65 | 94.8% | 0.91 | 1.25 | 1.26 | 95.5% |
| $\widehat{\beta}_3$ | −5.61 | 11.07 | 9.67 | 95.3 % | 0.04 | 1.10 | 1.14 | 94.9% |
| | $R_i \sim \mathrm{N}(0,1)$ | | | | $X_{ij} \sim \mathrm{N}(0.5,1)$ $Z_i \sim$ Poisson(0.5) | | | |
| $\widehat{\beta}_1$ | 6.38 | 12.62 | 9.60 | 93.8% | 0.17 | 1.11 | 1.08 | 95.1% |
| $\widehat{\beta}_2$ | 10.89 | 14.89 | 11.32 | 95.3% | 1.18 | 1.18 | 1.23 | 95.5% |
| $\widehat{\beta}_3$ | −8.63 | 13.57 | 10.27 | 95.6% | −0.42 | 1.04 | 1.11 | 96.4% |
| | $R_i \sim \mathrm{N}(0,1)$ | | | | $X_{ij} \sim \mathrm{N}(0.5,1)$ $Z_i \sim$ Geometric(0.7) | | | |
| $\widehat{\beta}_1$ | 4.87 | 8.17 | 7.33 | 95.5% | 0.89 | 0.97 | 1.04 | 96.9% |
| $\widehat{\beta}_2$ | 7.47 | 9.67 | 8.82 | 94.9% | 0.84 | 1.08 | 1.18 | 96.2% |
| $\widehat{\beta}_3$ | −4.18 | 8.47 | 7.57 | 94.3% | −0.06 | 1.04 | 1.06 | 95.5 % |

Table S.5: Simulation results when random effect and covariates are dependent: $X_{ij} \sim \mathrm{Normal}(0.5R_i, 1)$. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim 0.8\mathrm{N}(3,1) + 0.2\mathrm{N}(6,1.5)$ and centered | | | | $Z_i \sim$ Bernoulli(0.5) | | | |
| $\widehat{\beta}_1$ | 5.10 | 9.48 | 7.86 | 94.7% | 25.87 | 1.07 | 1.05 | 25.5% |
| $\widehat{\beta}_2$ | 10.29 | 11.49 | 9.51 | 94.6% | 25.09 | 1.27 | 1.21 | 37.5% |
| $\widehat{\beta}_3$ | −4.39 | 9.65 | 8.06 | 95.3% | 29.61 | 1.04 | 0.99 | 16.3% |
| | $R_i \sim 0.8\mathrm{N}(3,1) + 0.2\mathrm{N}(6,1.5)$ and centered | | | | $Z_i \sim$ Poisson(0.5) | | | |
| $\widehat{\beta}_1$ | 5.81 | 11.56 | 8.16 | 94.8% | 23.95 | 1.01 | 1.00 | 30.7% |
| $\widehat{\beta}_2$ | 9.93 | 14.54 | 9.75 | 94.5% | 23.19 | 1.17 | 1.16 | 40.0% |
| $\widehat{\beta}_3$ | −6.00 | 9.91 | 7.94 | 94.3% | 26.78 | 1.01 | 0.95 | 24.3% |
| | $R_i \sim 0.8\mathrm{N}(3,1) + 0.2\mathrm{N}(6,1.5)$ and centered | | | | $Z_i \sim$ Geometric(0.7) | | | |
| $\widehat{\beta}_1$ | 3.99 | 8.35 | 7.25 | 95.2% | 23.92 | 0.97 | 0.99 | 31.3% |
| $\widehat{\beta}_2$ | 7.50 | 10.29 | 8.46 | 93.9% | 22.95 | 1.16 | 1.14 | 44.9% |
| $\widehat{\beta}_3$ | −4.33 | 8.01 | 7.33 | 96.3% | 25.62 | 0.97 | 0.94 | 25.9% |

Table S.6: Simulation results when random effect and covariates are dependent: $X_{ij} \sim \text{Normal}(0.5R_i, 1)$. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim \text{Gamma}(1, 1.25)$ and centered | | | | $Z_i \sim \text{Bernoulli}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 5.24 | 10.3 | 7.92 | 93.2% | 22.41 | 0.97 | 0.97 | 36.1% |
| $\widehat{\beta}_2$ | 8.50 | 11.80 | 9.07 | 94.3% | 22.13 | 1.07 | 1.13 | 43.4% |
| $\widehat{\beta}_3$ | −5.87 | 10.38 | 8.14 | 94.5% | 24.45 | 0.87 | 0.91 | 29.0% |
| | $R_i \sim \text{Gamma}(1, 1.25)$ and centered | | | | $Z_i \sim \text{Poisson}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 5.40 | 9.58 | 8.01 | 94.9% | 21.32 | 1.00 | 0.94 | 40.4% |
| $\widehat{\beta}_2$ | 8.92 | 12.38 | 9.57 | 94.5% | 20.95 | 1.05 | 1.09 | 48.3% |
| $\widehat{\beta}_3$ | −7.29 | 11.20 | 8.54 | 92.9% | 23.31 | 0.91 | 0.90 | 31.6% |
| | $R_i \sim \text{Gamma}(1, 1.25)$ and centered | | | | $Z_i \sim \text{Geometric}(0.7)$ | | | |
| $\widehat{\beta}_1$ | 4.31 | 8.22 | 6.82 | 93.8% | 20.94 | 0.92 | 0.94 | 40.7% |
| $\widehat{\beta}_2$ | 8.24 | 10.96 | 8.1 | 93.4% | 20.89 | 1.06 | 1.09 | 49.0% |
| $\widehat{\beta}_3$ | −4.24 | 8.65 | 6.91 | 93.8% | 23.12 | 0.89 | 0.89 | 30.5% |

Table S.7: Simulation results when random effect and covariates are dependent: $X_{ij} \sim \text{Normal}(0.5R_i, 1)$. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim \text{t}(3)$ and centered | | | | $Z_i \sim \text{Bernoulli}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.90 | 10.22 | 8.97 | 94.6% | 34.99 | 1.13 | 1.13 | 7.1% |
| $\widehat{\beta}_2$ | 10.85 | 13.81 | 10.75 | 93.3% | 33.27 | 1.33 | 1.31 | 14.6% |
| $\widehat{\beta}_3$ | −7.22 | 10.33 | 9.24 | 95.7% | 39.86 | 0.92 | 0.98 | 2.5% |
| | $R_i \sim \text{t}(3)$ and centered | | | | $Z_i \sim \text{Poisson}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 5.08 | 11.66 | 9.18 | 93.7% | 32.12 | 1.02 | 1.03 | 10.0% |
| $\widehat{\beta}_2$ | 8.67 | 14.20 | 10.74 | 94.9% | 30.26 | 1.27 | 1.21 | 17.1% |
| $\widehat{\beta}_3$ | −8.05 | 14.84 | 9.92 | 94.2% | 37.37 | 0.85 | 0.90 | 3.4% |
| | $R_i \sim \text{t}(3)$ and centered | | | | $Z_i \sim \text{Geometric}(0.7)$ | | | |
| $\widehat{\beta}_1$ | 4.54 | 11.4 | 8.73 | 94.3% | 31.47 | 0.96 | 1.03 | 10.4% |
| $\widehat{\beta}_2$ | 8.47 | 13.4 | 10.31 | 94.5% | 29.99 | 1.26 | 1.21 | 18.6% |
| $\widehat{\beta}_3$ | −4.95 | 10.10 | 8.87 | 94.7% | 36.89 | 0.96 | 0.91 | 4.0% |

Table S.8: Simulation results when random effect and covariates are dependent: $X_{ij} \sim \text{Normal}(0.5R_i, 1)$. Bias, sample variance (var), averaged estimated variance ($\widehat{\text{var}}$), and the empirical coverage percentage of the 95% confidence interval (CI) for the semiparametric estimator and the normal-based MLE are reported. The true parameter $\boldsymbol{\beta} = (0.35, 0.6, -0.4)^{\mathrm{T}}$. Results are based on 1000 simulations with $n = 220$, $m_i = 3$. Biases are multiplied by 100, var and $\widehat{\text{var}}$ are multiplied by 1000.

| | Semiparametric Estimator | | | | Normal-based MLE | | | |
|---|---|---|---|---|---|---|---|---|
| | bias | var | $\widehat{\text{var}}$ | CI | bias | var | $\widehat{\text{var}}$ | CI |
| | $R_i \sim \text{N}(0,1)$ | | | | $Z_i \sim \text{Bernoulli}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 4.74 | 12.41 | 11.37 | 95.1% | 30.59 | 1.1 | 1.1 | 13.6% |
| $\widehat{\beta}_2$ | 10.64 | 14.65 | 13.99 | 95.3% | 29.60 | 1.37 | 1.27 | 23.1% |
| $\widehat{\beta}_3$ | −6.51 | 11.13 | 10.90 | 96.5% | 35.75 | 1.03 | 1.01 | 6.8% |
| | $R_i \sim \text{N}(0,1)$ | | | | $Z_i \sim \text{Poisson}(0.5)$ | | | |
| $\widehat{\beta}_1$ | 8.03 | 18.74 | 13.53 | 94.3% | 29.71 | 1.06 | 1.06 | 15.2% |
| $\widehat{\beta}_2$ | 12.46 | 25.42 | 15.66 | 94.7% | 28.17 | 1.18 | 1.22 | 25.1% |
| $\widehat{\beta}_3$ | −9.40 | 15.52 | 11.69 | 94.9% | 34.50 | 0.98 | 0.98 | 8.2% |
| | $R_i \sim \text{N}(0,1)$ | | | | $Z_i \sim \text{Geometric}(0.7)$ | | | |
| $\widehat{\beta}_1$ | 5.65 | 10.27 | 8.38 | 95.4% | 28.86 | 1.03 | 1.02 | 17.2% |
| $\widehat{\beta}_2$ | 8.45 | 12.51 | 9.86 | 94.8% | 26.80 | 1.19 | 1.18 | 28.6% |
| $\widehat{\beta}_3$ | −5.63 | 9.74 | 8.58 | 95.3% | 33.43 | 1.02 | 0.94 | 10.0% |