Prediction Error of Small Area Predictors Shrinking both Means and Variances

Supplementary Appendix

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Appendix S1 Estimation of the parameters

This section provides the detailed calculations used for the EM algorithm in Subsection 3.2. The logarithm of the complete data likelihood is

$$\begin{aligned}\log(L_{\text{compl}}) &= \sum_{i=1}^{n}\Big[\log(\text{Constant}_i) + \log\{\Gamma(\frac{n_i}{2}+\alpha)\} - \log\{\Gamma(\alpha)\} - \frac{1}{2}\log\tau^2 - \\ &\quad \alpha\log(\gamma) - \frac{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2}{2\tau^2} - \Big(\frac{n_i}{2}+\alpha\Big)\log(\psi_i)\Big],\end{aligned}$$

and in the E-step of the t^{th} iteration we calculate

$$\begin{aligned}Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)}) &\equiv E^{(t-1)}\{\log(L_{\text{compl}})\} = \sum_{i=1}^{n}\Big[\log(\text{Constant}_i) + \log\{\Gamma(\frac{n_i}{2}+\alpha)\} - \log\{\Gamma(\alpha)\} \\ &\quad -\frac{1}{2}\log\tau^2 - \alpha\log(\gamma) - E^{(t-1)}\Big\{\frac{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2}{2\tau^2}\Big\} - \Big(\frac{n_i}{2}+\alpha\Big)E^{(t-1)}\{\log(\psi_i)\}\Big],\end{aligned}$$

where the expectation is with respect to the conditional density of θ_i given in equation (5) with the parameters from $(t-1)^{th}$ iteration.

In the M-step of the t^{th} iteration we determine $\boldsymbol{\beta}$, τ^2, α and γ by maximizing Q, and call them as $\hat{\boldsymbol{\beta}}^{(t)}$, $\hat{\tau^2}^{(t)}$, $\hat{\alpha}^{(t)}$, and $\hat{\gamma}^{(t)}$. By setting

$$\frac{\partial Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)})}{\partial \boldsymbol{\beta}} = 0, \; \frac{\partial Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)})}{\partial \tau^2} = 0,$$

we obtain the expressions for $\hat{\boldsymbol{\beta}}^{(t)}$ and $\hat{\tau^2}$ given in Subsection 3.2. In particular,

$$\begin{aligned}\frac{\partial Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)})}{\partial \boldsymbol{\beta}} &= \sum_{i=1}^{n} \left[-\frac{\partial}{\partial \boldsymbol{\beta}} E^{(t-1)} \Big\{ \frac{(\theta_i - \boldsymbol{Z}_i^T \boldsymbol{\beta})^2}{2\tau^2} \Big\} - \left(\frac{n_i}{2} + \alpha \right) \frac{\partial}{\partial \boldsymbol{\beta}} E^{(t-1)} \{ \log(\psi_i) \} \right] \\ &= \sum_{i=1}^{n} E^{(t-1)} \Big\{ \frac{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T \boldsymbol{\beta})}{\tau^2} \Big\}.\end{aligned}$$

Similarly, α and γ are estimated by solving $S_\alpha = 0$ and $S_\gamma = 0$ where

$$S_\alpha = \frac{\partial Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)})}{\partial \alpha} = \sum_{i=1}^{n} \Big[\log'\{\Gamma(\frac{n_i}{2} + \alpha)\} - \log'\{\Gamma(\alpha)\} - \log(\gamma) - E^{(t-1)}\{\log(\psi_i)\} \Big] \quad (1)$$

$$S_\gamma = \frac{\partial Q(\boldsymbol{B}|\boldsymbol{B}^{(t-1)})}{\partial \gamma} = -\frac{n\alpha}{\gamma} + \frac{1}{\gamma^2} \sum_{i=1}^{n} (\frac{n_i}{2} + \alpha) E^{(t-1)} \left(\frac{1}{\psi_i} \right). \quad (2)$$

In order to solve $S_\alpha = 0$ and $S_\gamma = 0$, we require the following components

$$\begin{aligned} S_{\alpha\alpha} &= \textstyle\sum_{i=1}^{n} \left[\log''\{\Gamma(\frac{n_i}{2} + \alpha)\} - \log''\{\Gamma(\alpha)\} \right] \\ S_{\alpha\gamma} &= \textstyle\sum_{i=1}^{n} \left\{ -\frac{1}{\gamma} + \frac{1}{\gamma^2} E^{(t-1)} \left(\frac{1}{\psi_i} \right) \right\} \\ S_{\gamma\alpha} &= S_{\alpha\gamma} \\ S_{\gamma\gamma} &= \textstyle\sum_{i=1}^{n} \left\{ \frac{\alpha}{\gamma^2} - (n_i + 2\alpha) \frac{1}{\gamma^3} E^{(t-1)} \left(\frac{1}{\psi_i} \right) + (\frac{n_i}{2} + \alpha) \frac{1}{\gamma^4} E^{(t-1)} \left(\frac{1}{\psi_i^2} \right) \right\}, \end{aligned} \quad (3)$$

and then α and γ are estimated by the Newton-Raphson method:

$$\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}^{l} = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}^{(l-1)} - \begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\gamma} \\ S_{\gamma\alpha} & S_{\gamma\gamma} \end{bmatrix}^{-1} \begin{bmatrix} S_\alpha \\ S_\gamma \end{bmatrix}.$$

Appendix S2 Computation of $\hat{\boldsymbol{B}} - \boldsymbol{B}$

In this section of appendix, the detail expression of equation (12) is given. For the first order derivative terms, $U_r^{(i)}$ is defined as

$$U_r^{(i)} = \frac{\partial \log L_i^M}{\partial \boldsymbol{B}_r}$$

where the marginal log-likelihood is given in the section 3.2. In particular, the partial derivative with respect to α is

$$
\begin{aligned}
U_\alpha^{(i)} &= \frac{\partial \log L_i^M}{\partial \alpha} = E\left(\frac{\partial \log L_{i,\text{compl}}}{\partial \alpha}\right) \\
&= \log'\{\Gamma(\frac{n_i}{2}+\alpha)\} - \log'\{\Gamma(\alpha)\} - \log(\gamma) - E\{\log(\psi_i)\}
\end{aligned}
$$

where expectation is with respect to the conditional distribution of θ_i, and $\log(L_{i,\text{compl}})$ is given in appendix A. The other terms are

$$
\begin{aligned}
U_\gamma^{(i)} &= -\frac{\alpha}{\gamma} + \frac{1}{\gamma^2}(\frac{n_i}{2}+\alpha)E\left(\frac{1}{\psi_i}\right), \\
U_{\boldsymbol{\beta}}^{(i)} &= E\left\{\frac{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})}{\tau^2}\right\}, \\
U_{\tau^2}^{(i)} &= -\frac{1}{2\tau^2} + E\left\{\frac{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2}{2(\tau^2)^2}\right\}.
\end{aligned}
$$

The second order derivative with respect to α is

$$
\begin{aligned}
V_{\alpha\alpha}^{(i)} &= \frac{\partial U_\alpha^{(i)}}{\partial \alpha} = \frac{\partial}{\partial \alpha} E(\partial \log L_{i,\text{compl}}/\partial \alpha) \\
&= E\left(\frac{\partial^2 \log\{L_{i,\text{compl}}\}}{\partial \alpha^2}\right) + E\left\{\left(\frac{\partial \log\{L_{i,\text{compl}}\}}{\partial \alpha}\right)^2\right\} - E^2\left\{\left(\frac{\partial \log\{L_{i,\text{compl}}\}}{\partial \alpha}\right)\right\} \\
&= E\left(\frac{\partial^2 \log\{L_{i,\text{compl}}\}}{\partial \alpha^2}\right) + \text{var}\left\{\left(\frac{\partial \log\{L_{i,\text{compl}}\}}{\partial \alpha}\right)\right\} \\
&= \log''\{\Gamma(\frac{n_i}{2}+\alpha)\} - \log''\{\Gamma(\alpha)\} + \text{var}\{\log(\psi_i)\}
\end{aligned}
$$

The other second order derivative terms are

$$
\begin{aligned}
V_{\gamma\gamma}^{(i)} &= \frac{\alpha}{\gamma^2} - (n_i+2\alpha)\frac{1}{\gamma^3}E\left(\frac{1}{\psi_i}\right) + (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^4}E\left(\frac{1}{\psi_i^2}\right) + (\frac{n_i}{2}+\alpha)^2\frac{1}{\gamma^4}\text{var}\left(\frac{1}{\psi_i}\right), \\
V_{\boldsymbol{\beta}\boldsymbol{\beta}}^{(i)} &= -\frac{\boldsymbol{Z}_i\boldsymbol{Z}_i^T}{\tau^2} + \frac{1}{(\tau^2)^2}\text{var}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}, \\
V_{\tau^2\tau^2}^{(i)} &= \frac{1}{2(\tau^2)^2} - \frac{1}{(\tau^2)^3}E(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2 + \frac{1}{4(\tau^2)^4}\text{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}.
\end{aligned}
$$

The other cross terms of the second order derivative are

$$
V_{\alpha\gamma}^{(i)} = -\frac{1}{\gamma} + \frac{1}{\gamma^2}E\left(\frac{1}{\psi_i}\right) - (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2}\text{cov}\left\{\frac{1}{\psi_i}, \log(\psi_i)\right\},
$$

$$
\begin{aligned}
V_{\alpha\boldsymbol{\beta}}^{(i)} &= -\frac{1}{\tau^2}\text{cov}\{\log(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}, \\
V_{\alpha\tau^2}^{(i)} &= -\frac{1}{2(\tau^2)^2}\text{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}, \\
V_{\gamma\boldsymbol{\beta}}^{(i)} &= (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2\tau^2}\text{cov}\{\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}, \\
V_{\gamma\tau^2}^{(i)} &= (\frac{n_i}{2}+\alpha)\frac{1}{2\gamma^2\tau^4}\text{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}, \\
V_{\boldsymbol{\beta}\tau^2}^{(i)} &= -\frac{1}{\tau^4}E\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} + \frac{1}{2\tau^6}\text{cov}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}.
\end{aligned}
$$

The third derivative terms are:

$$
\begin{aligned}
W_{\alpha\alpha\alpha}^{(i)} &= \log\Gamma'''(\frac{n_i}{2}+\alpha) - \log\Gamma'''(\alpha) - \text{cov}\{\log(\psi_i), \log^2(\psi_i)\} + 2E\log(\psi_i)\text{var}\{\log(\psi_i)\}, \\
W_{\gamma\gamma\gamma}^{(i)} &= -\frac{2\alpha}{\gamma^3} + (\frac{n_i}{2}+\alpha)\left(\frac{6}{\gamma^4}E\frac{1}{\psi_i} - \frac{6}{\gamma^5}E\frac{1}{\psi_i^2} + \frac{2}{\gamma^6}E\frac{1}{\psi_i^3}\right) - (\frac{n_i}{2}+\alpha)^2\left\{\frac{6}{\gamma^5}\text{var}(\frac{1}{\psi_i}) + \right. \\
&\left. \frac{1}{\gamma^6}\text{cov}(\frac{1}{\psi_i}, \frac{1}{\psi_i^2})\right\} - (\frac{n_i}{2}+\alpha)^3\frac{1}{\gamma^6}\left\{\text{cov}(\frac{1}{\psi_i}, \frac{1}{\psi_i^2}) - 2E\frac{1}{\psi_i}\text{var}(\frac{1}{\psi_i})\right\}, \\
W_{\boldsymbol{\beta\beta\beta}}^{(i)} &= \frac{1}{(\tau^2)^3}\left[\text{cov}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - 2E(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\text{var}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\right], \\
W_{\tau^2\tau^2\tau^2}^{(i)} &= -\frac{1}{(\tau^2)^3} + \frac{3}{(\tau^2)^4}E(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2 - \frac{3}{2(\tau^2)^5}\text{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} + \frac{1}{8(\tau^2)^6}\cdot \\
&\times\left[\text{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^4, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - 2E(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\text{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right].
\end{aligned}
$$

The other cross terms of the third order derivative are

$$
\begin{aligned}
W_{\alpha\alpha\gamma}^{(i)} &= (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2}\left[\text{cov}\{\log^2(\psi_i), \frac{1}{\psi_i}\} - 2E\log(\psi_i)\text{cov}\{\log(\psi_i), \frac{1}{\psi_i}\}\right] - \\
&\frac{2}{\gamma^2}\text{cov}\{\log(\psi_i), \frac{1}{\psi_i}\}, \\
W_{\alpha\alpha\boldsymbol{\beta}}^{(i)} &= \frac{1}{\tau^2}\left[\text{cov}\{\log^2(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} - 2E\log(\psi_i)\text{cov}\{\log(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}\right], \\
W_{\alpha\alpha\tau^2}^{(i)} &= \frac{1}{2(\tau^2)^2}\left[\text{cov}\{\log^2(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - 2E\log(\psi_i)\text{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right], \\
W_{\alpha\gamma\gamma}^{(i)} &= \frac{1}{\gamma^2} - \frac{2}{\gamma^3}E\frac{1}{\psi_i} - \frac{1}{\gamma^4}E\frac{1}{\psi_i^2} + (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^4}\left[2\text{var}(\frac{1}{\psi_i}) - \text{cov}\{\log(\psi_i), \frac{1}{\psi_i^2}\}\right] - \\
&(\frac{n_i}{2}+\alpha)^2\frac{1}{\gamma^4}\left[\text{cov}\{\log(\psi_i)\cdot\frac{1}{\psi_i}, \frac{1}{\psi_i}\} - \text{cov}\{\log(\psi_i), \frac{1}{\psi_i}\}E\frac{1}{\psi_i} - \right. \\
&\left. E\log(\psi_i)\text{var}(\frac{1}{\psi_i})\right],
\end{aligned}
$$

$$
W_{\alpha\gamma\boldsymbol{\beta}}^{(i)} = \frac{1}{\gamma^2\tau^2}\text{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) - (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2\tau^2}\left[\text{cov}\{\log(\psi_i)\cdot\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} - \right.
$$

$$
\begin{aligned}
& \operatorname{cov}\{\log(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}E\frac{1}{\psi_i} - E\log(\psi_i)\operatorname{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\Big], \\
W^{(i)}_{\alpha\gamma\tau^2} &= \frac{1}{2\gamma^2(\tau^2)^2}\operatorname{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - (\frac{n_i}{2}+\alpha)\frac{1}{2\gamma^2(\tau^2)^2}\Big[\operatorname{cov}\{\log(\psi_i)\cdot\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \\
& \operatorname{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\frac{1}{\psi_i} - E\log(\psi_i)\operatorname{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\Big], \\
W^{(i)}_{\alpha\boldsymbol{\beta}\boldsymbol{\beta}} &= -\frac{1}{(\tau^2)^2}\left[\operatorname{cov}\{\log(\psi_i), (\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))^2\} - 2E(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\operatorname{cov}\{\log(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}\right], \\
W^{(i)}_{\alpha\boldsymbol{\beta}\tau^2} &= \frac{1}{(\tau^2)^2}\operatorname{cov}\{\log(\psi_i), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} - \frac{1}{2(\tau^2)^3}\left[\operatorname{cov}\{\log(\psi_i)\cdot\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \right. \\
& \left.\operatorname{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\boldsymbol{Z}_i((\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\log(\psi_i)\operatorname{cov}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right], \\
W^{(i)}_{\alpha\tau^2\tau^2} &= \frac{1}{4(\tau^2)^3}\operatorname{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \frac{1}{4(\tau^2)^4}\left[\operatorname{cov}\{\log(\psi_i)\cdot(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \right. \\
& \left.\operatorname{cov}\{\log(\psi_i), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2 - E\log(\psi_i)\operatorname{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right], \\
W^{(i)}_{\gamma\gamma\boldsymbol{\beta}} &= -(\frac{n_i}{2}+\alpha)\frac{2}{\gamma^3\tau^2}\operatorname{cov}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), \frac{1}{\psi_i}) + (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^4\tau^2}\operatorname{cov}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), \frac{1}{\psi_i^2}) + \\
& (\frac{n_i}{2}+\alpha)^2\frac{1}{\gamma^4\tau^2}\left\{\operatorname{cov}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), \frac{1}{\psi_i^2}) - 2E\frac{1}{\psi_i}\operatorname{cov}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), \frac{1}{\psi_i})\right\}, \\
W^{(i)}_{\gamma\gamma\tau^2} &= -(\frac{n_i}{2}+\alpha)\frac{1}{\gamma^3(\tau^2)^2}\operatorname{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \frac{1}{\psi_i}\} + (\frac{n_i}{2}+\alpha)\frac{1}{2\gamma^4(\tau^2)^2}\operatorname{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \frac{1}{\psi_i^2}\} + \\
& (\frac{n_i}{2}+\alpha)^2\frac{1}{2\gamma^4(\tau^2)^2}\left[\operatorname{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \frac{1}{\psi_i^2}\} - 2E\frac{1}{\psi_i}\operatorname{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \frac{1}{\psi_i}\}\right], \\
W^{(i)}_{\gamma\boldsymbol{\beta}\boldsymbol{\beta}} &= (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2(\tau^2)^2}\Big[\operatorname{cov}\{\frac{1}{\psi_i}\cdot\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} - \operatorname{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\cdot \\
& E\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\frac{1}{\psi_i}\operatorname{var}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\Big] \\
W^{(i)}_{\gamma\boldsymbol{\beta}\tau^2} &= -(\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2(\tau^2)^2}\operatorname{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) + (\frac{n_i}{2}+\alpha)\frac{1}{2\gamma^2(\tau^2)^3}\cdot \\
& \Big[\operatorname{cov}\{\frac{1}{\psi_i}\cdot\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \operatorname{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}) \\
& -E\frac{1}{\psi_i}\operatorname{cov}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}), (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\Big] \\
W^{(i)}_{\gamma\tau^2\tau^2} &= -(\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2(\tau^2)^3}\operatorname{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} + (\frac{n_i}{2}+\alpha)\frac{1}{4\gamma^2(\tau^2)^4}\cdot \\
& \Big[\operatorname{cov}\{\frac{1}{\psi_i}\cdot(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \operatorname{cov}\{\frac{1}{\psi_i}, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2 \\
& -E\frac{1}{\psi_i}\operatorname{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\Big] \\
W^{(i)}_{\boldsymbol{\beta}\boldsymbol{\beta}\tau^2} &= \frac{\boldsymbol{Z}_i\boldsymbol{Z}_i^T}{(\tau^2)^2} - \frac{2}{(\tau^2)^3}\operatorname{var}(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) + \frac{1}{2(\tau^2)^4}\left[\operatorname{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right.
\end{aligned}
$$

$$
\begin{aligned}
& -2E\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\mathrm{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\}\big] \\
W^{(i)}_{\boldsymbol{\beta}\tau^2\tau^2} \quad = \quad & \frac{2}{(\tau^2)^3}E\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}) - \frac{2}{(\tau^2)^4}\mathrm{cov}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})\} + \frac{1}{4(\tau^2)^5}\cdot \\
& \big[\mathrm{cov}\{\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^3, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \mathrm{var}\{(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E(\boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\big]
\end{aligned}
$$

All the other terms are equal to the above values according to symmetry.

Appendix S3 Second order correction of ν_i

In the equation (11), the second order derivatives of ν_i are included. The detail calculations are given as follow:

$$
\begin{aligned}
\frac{\partial^2\nu_i}{\partial\alpha^2} &= -\frac{\partial}{\partial\alpha}\mathrm{cov}(\theta_i^2, \log\psi_i) + 2\frac{\partial}{\partial\alpha}E\theta_i\cdot\mathrm{cov}(\theta_i, \log\psi_i) + 2E\theta_i\frac{\partial}{\partial\alpha}\mathrm{cov}(\theta_i, \log\psi_i) \\
\frac{\partial}{\partial\alpha}\mathrm{cov}(\theta_i^2, \log\psi_i) &= -\mathrm{cov}(\theta_i^2\log\psi_i, \log\psi_i) + \mathrm{cov}(\theta_i^2, \log\psi_i)E\log\psi_i + E\theta_i^2\mathrm{cov}(\log\psi_i, \log\psi_i) \\
\frac{\partial}{\partial\alpha}E\theta_i &= -\mathrm{cov}(\theta_i, \log\psi_i) \\
\frac{\partial}{\partial\alpha}\mathrm{cov}(\theta_i, \log\psi_i) &= -\mathrm{cov}(\theta_i\log\psi_i, \log\psi_i) + \mathrm{cov}(\theta_i, \log\psi_i)E\log\psi_i + E\theta_i\mathrm{cov}(\log\psi_i, \log\psi_i)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\nu_i}{\partial\alpha\partial\gamma} &= -\frac{\partial}{\partial\gamma}\mathrm{cov}(\theta_i^2, \log\psi_i) + 2\frac{\partial}{\partial\gamma}E\theta_i\cdot\mathrm{cov}(\theta_i, \log\psi_i) + 2E\theta_i\frac{\partial}{\partial\gamma}\mathrm{cov}(\theta_i, \log\psi_i) \\
\frac{\partial}{\partial\gamma}\mathrm{cov}(\theta_i^2, \log\psi_i) &= (n_i/2+\alpha)\frac{1}{\gamma^2}\left\{\mathrm{cov}(\theta_i^2\log\psi_i, \frac{1}{\psi_i}) - \mathrm{cov}(\theta_i^2, \frac{1}{\psi_i})E\log\psi_i - E\theta_i^2\mathrm{cov}(\log\psi_i, \frac{1}{\psi_i})\right\} - \\
&\quad \frac{1}{\gamma^2}\mathrm{cov}(\theta_i^2, \frac{1}{\psi_i}) \\
\frac{\partial}{\partial\gamma}E\theta_i &= (n_i/2+\alpha)\frac{1}{\gamma^2}\mathrm{cov}(\theta_i, \frac{1}{\psi_i}) \\
\frac{\partial}{\partial\gamma}\mathrm{cov}(\theta_i, \log\psi_i) &= (n_i/2+\alpha)\frac{1}{\gamma^2}\left\{\mathrm{cov}(\theta_i\log\psi_i, \frac{1}{\psi_i}) - \mathrm{cov}(\theta_i, \frac{1}{\psi_i})E\log\psi_i - E\theta_i\mathrm{cov}(\log\psi_i, \frac{1}{\psi_i})\right\} - \\
&\quad \frac{1}{\gamma^2}\mathrm{cov}(\theta_i, \frac{1}{\psi_i})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\nu_i}{\partial\alpha\partial\boldsymbol{\beta}} &= -\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i^2, \log\psi_i) + 2\frac{\partial}{\partial\boldsymbol{\beta}}E\theta_i\cdot\mathrm{cov}(\theta_i, \log\psi_i) + 2E\theta_i\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i, \log\psi_i) \\
\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i^2, \log\psi_i) &= \frac{1}{\tau^2}\left\{\mathrm{cov}(\theta_i^2\log\psi_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) - \mathrm{cov}(\theta_i^2, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))E\log\psi_i\right. \\
&\quad \left.-E\theta_i^2\mathrm{cov}(\log\psi_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\beta}} E\theta_i &= \frac{\boldsymbol{Z}_i}{\tau^2}\text{var}(\theta_i) \\
\frac{\partial}{\partial \boldsymbol{\beta}}\text{cov}(\theta_i, \text{log}\psi_i) &= \frac{1}{\tau^2}\left\{\text{cov}(\theta_i\text{log}\psi_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) - \text{cov}(\theta_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))E\text{log}\psi_i \right. \\
&\quad \left. - E\theta_i\text{cov}(\text{log}\psi_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\alpha\partial\tau^2} &= -\frac{\partial}{\partial\tau^2}\text{cov}(\theta_i^2, \text{log}\psi_i) + 2\frac{\partial}{\partial\tau^2}E\theta_i \cdot \text{cov}(\theta_i, \text{log}\psi_i) + 2E\theta_i\frac{\partial}{\partial\tau^2}\text{cov}(\theta_i, \text{log}\psi_i) \\
\frac{\partial}{\partial\tau^2}\text{cov}(\theta_i^2, \text{log}\psi_i) &= \frac{1}{2(\tau^2)^2}\left[\text{cov}\{\theta_i^2\text{log}\psi_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \text{cov}\{\theta_i^2, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\text{log}\psi_i - \right. \\
&\quad \left. E\theta_i^2\text{cov}\{\text{log}\psi_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right] \\
\frac{\partial}{\partial\tau^2}E\theta_i &= \frac{1}{2(\tau^2)^2}\text{cov}\{\theta_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
\frac{\partial}{\partial\tau^2}\text{cov}(\theta_i, \text{log}\psi_i) &= \frac{1}{2(\tau^2)^2}\left[\text{cov}\{\theta_i\text{log}\psi_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \text{cov}\{\theta_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\text{log}\psi_i - \right. \\
&\quad \left. E\theta_i\text{cov}\{\text{log}\psi_i, (\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\gamma^2} &= -(\frac{n_i}{2} + \alpha)\frac{2}{\gamma^3}\left\{\text{cov}(\theta_i^2, \frac{1}{\psi_i}) - 2E\theta_i\text{cov}(\theta_i, \frac{1}{\psi_i})\right\} + (\frac{n_i}{2} + \alpha)\frac{1}{\gamma^2}\left\{\frac{\partial}{\partial\gamma}\text{cov}(\theta_i^2, \frac{1}{\psi_i}) - \right. \\
&\quad \left. 2\frac{\partial}{\partial\gamma}E\theta_i \cdot \text{cov}(\theta_i, \frac{1}{\psi_i}) - 2E\theta_i\frac{\partial}{\partial\gamma}\text{cov}(\theta_i, \frac{1}{\psi_i})\right\} \\
\frac{\partial}{\partial\gamma}\text{cov}(\theta_i^2, \frac{1}{\psi_i}) &= (\frac{n_i}{2} + \alpha)\frac{1}{\gamma^2}\left\{\text{cov}(\theta_i^2\frac{1}{\psi_i}, \frac{1}{\psi_i}) - \text{cov}(\theta_i^2, \frac{1}{\psi_i})E\frac{1}{\psi_i} - E\theta_i^2\text{var}(\frac{1}{\psi_i})\right\} + \frac{1}{\gamma^2}\text{cov}(\theta_i^2, \frac{1}{\psi_i^2}) \\
\frac{\partial}{\partial\gamma}E\theta_i &= (n_i/2 + \alpha)\frac{1}{\gamma^2}\text{cov}(\theta_i, \frac{1}{\psi_i}) \\
\frac{\partial}{\partial\gamma}\text{cov}(\theta_i, \frac{1}{\psi_i}) &= (\frac{n_i}{2} + \alpha)\frac{1}{\gamma^2}\left\{\text{cov}(\theta_i\frac{1}{\psi_i}, \frac{1}{\psi_i}) - \text{cov}(\theta_i, \frac{1}{\psi_i})E\frac{1}{\psi_i} - E\theta_i\text{var}(\frac{1}{\psi_i})\right\} + \frac{1}{\gamma^2}\text{cov}(\theta_i, \frac{1}{\psi_i^2})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\gamma\partial\boldsymbol{\beta}} &= (\frac{n_i}{2} + \alpha)\frac{1}{\gamma^2}\left\{\frac{\partial}{\partial\boldsymbol{\beta}}\text{cov}(\theta_i^2, \frac{1}{\psi_i}) - 2\frac{\partial}{\partial\boldsymbol{\beta}}E\theta_i \cdot \text{cov}(\theta_i, \frac{1}{\psi_i}) - 2E\theta_i\frac{\partial}{\partial\boldsymbol{\beta}}\text{cov}(\theta_i, \frac{1}{\psi_i})\right\} \\
\frac{\partial}{\partial\boldsymbol{\beta}}\text{cov}(\theta_i^2, \frac{1}{\psi_i}) &= \frac{1}{\tau^2}\left\{\text{cov}(\theta_i^2\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) - \text{cov}(\theta_i^2, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))E\frac{1}{\psi_i} \right. \\
&\quad \left. - E\theta_i^2\text{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\} \\
\frac{\partial}{\partial\boldsymbol{\beta}}E\theta_i &= \frac{\boldsymbol{Z}_i}{\tau^2}\text{var}(\theta_i) \\
\frac{\partial}{\partial\boldsymbol{\beta}}\text{cov}(\theta_i, \frac{1}{\psi_i}) &= \frac{1}{\tau^2}\left\{\text{cov}(\theta_i\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta})) - \text{cov}(\theta_i, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))E\frac{1}{\psi_i} \right. \\
&\quad \left. - E\theta_i\text{cov}(\frac{1}{\psi_i}, \boldsymbol{Z}_i(\theta_i - \boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\gamma\partial\tau^2} &= (\frac{n_i}{2}+\alpha)\frac{1}{\gamma^2}\left\{\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i^2,\frac{1}{\psi_i}) - 2\frac{\partial}{\partial\tau^2}E\theta_i \times \mathrm{cov}(\theta_i,\frac{1}{\psi_i}) - 2E\theta_i\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i,\frac{1}{\psi_i})\right\} \\
\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i^2,\frac{1}{\psi_i}) &= \frac{1}{2(\tau^2)^2}\Big[\mathrm{cov}\{\theta_i^2\frac{1}{\psi_i},(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \mathrm{cov}\{\theta_i^2,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\frac{1}{\psi_i} \\
&\quad -E\theta_i^2\mathrm{cov}\{\frac{1}{\psi_i},(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\Big] \\
\frac{\partial}{\partial\tau^2}E\theta_i &= \frac{1}{2(\tau^2)^2}\mathrm{cov}\{\theta_i,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i,\frac{1}{\psi_i}) &= \frac{1}{2(\tau^2)^2}\Big[\mathrm{cov}\{\theta_i\frac{1}{\psi_i},(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \mathrm{cov}\{\theta_i,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}E\frac{1}{\psi_i} \\
&\quad -E\theta_i\mathrm{cov}\{\frac{1}{\psi_i},(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\boldsymbol{\beta}^2} &= \frac{1}{\tau^2}\left\{\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i^2,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) - 2\frac{\partial}{\partial\boldsymbol{\beta}}E\theta_i\cdot\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\right. \\
&\quad \left. -2E\theta_i\cdot\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\} \\
\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i^2,\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}) &= \frac{1}{\tau^2}\big[\mathrm{cov}\{\theta_i^2\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})\} - \mathrm{cov}(\theta_i^2,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) \\
&\quad \times E\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\theta_i^2\mathrm{var}(\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\big] \\
\frac{\partial}{\partial\boldsymbol{\beta}}E\theta_i &= \mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) \\
\frac{\partial}{\partial\boldsymbol{\beta}}\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) &= \frac{1}{\tau^2}\big[\mathrm{cov}\{\theta_i\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})\} - \mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) \\
&\quad \times E\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\theta_i\mathrm{var}(\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial\boldsymbol{\beta}\partial\tau^2} &= -\frac{1}{(\tau^2)^2}\left\{\mathrm{cov}(\theta_i^2,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) - 2E\theta_i\times\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\} \\
&\quad +\frac{1}{\tau^2}\left\{\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i^2,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) - 2\frac{\partial}{\partial\tau^2}E\theta_i\times\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\right. \\
&\quad \left. -2E\theta_i\cdot\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}))\right\} \\
\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i^2,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) &= \frac{1}{2(\tau^2)^2}\big[\mathrm{cov}\{\theta_i^2\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \mathrm{cov}\{\theta_i^2,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
&\quad \times E\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\theta_i^2\mathrm{cov}\{\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\big] \\
\frac{\partial}{\partial\tau^2}E\theta_i &= \frac{1}{2(\tau^2)^2}\mathrm{cov}\{\theta_i,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
\frac{\partial}{\partial\tau^2}\mathrm{cov}(\theta_i,\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})) &= \frac{1}{2(\tau^2)^2}\big[\mathrm{cov}\{\theta_i\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \mathrm{cov}\{\theta_i,(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
&\quad \times E\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}) - E\theta_i\mathrm{cov}\{\boldsymbol{Z}_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta}),(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \nu_i}{\partial {\tau^2}^2} &= -\frac{1}{(\tau^2)^3}\left[\text{cov}\{\theta_i^2, (\theta_i-\boldsymbol{\beta})^2\} - 2E\theta_i \text{cov}\{\theta_i, (\theta_i-\boldsymbol{\beta})^2\}\right] + \frac{1}{2(\tau^2)^2} \\
&\quad \times \left[\frac{\partial}{\partial \tau^2}\text{cov}\{\theta_i^2, (\theta_i-\boldsymbol{\beta})^2\} - 2\frac{\partial}{\partial \tau^2}E\theta_i \times \text{cov}\{\theta_i, (\theta_i-\boldsymbol{\beta})^2\} - \right. \\
&\quad \left. 2E\theta_i \times \frac{\partial}{\partial \tau^2}\text{cov}\{\theta_i, (\theta_i-\boldsymbol{\beta})^2\}\right], \\
\frac{\partial}{\partial \tau^2}\text{cov}\{\theta_i^2, (\theta_i-\boldsymbol{\beta})^2\} &= \frac{1}{2(\tau^2)^2}\left[\text{cov}\{\theta_i^2(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \text{cov}\{\theta_i^2, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right. \\
&\quad \left. \times E(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2 - E\theta_i^2 \text{var}\{(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right], \\
\frac{\partial}{\partial \tau^2}E\theta_i &= \frac{1}{2(\tau^2)^2}\text{cov}\{\theta_i, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} \\
\frac{\partial}{\partial \tau^2}\text{cov}\{\theta_i, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} &= \frac{1}{2(\tau^2)^2}\left[\text{cov}\{\theta_i(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\} - \text{cov}\{\theta_i, (\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right. \\
&\quad \left. \times E(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2 - E\theta_i \text{var}\{(\theta_i-\boldsymbol{Z}_i^T\boldsymbol{\beta})^2\}\right].
\end{aligned}
$$

The other terms are obtained using symmetry property.

Appendix S4 Information Matrix for $\boldsymbol{B}$

The Fisher information matrix is $I_{sr} = -E\left[\partial^2 \text{log-likelihood}(X_i, S_i^2)/\partial \boldsymbol{B}_s \partial \boldsymbol{B}_r\right]$, and the observed information matrix is $-n^{-1}\sum_{i=1}^n \partial^2 \text{log-likelihood}(X_i, S_i^2)/\partial \boldsymbol{B}_s \partial \boldsymbol{B}_r$. The terms of this matrix are obtained from appendix B.